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GUTs and exceptional branes in F-theory — II. Experimental predictions

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Abstract: We consider realizations of GUT models in F-theory. Adopting a bottom up approach, the assumption that the dynamics of the GUT model can in principle decouple from Planck scale physics leads to a surprisingly predictive framework. An internal U(1) hypercharge flux Higgses the GUT group directly to the MSSM or to a flipped GUT model, a mechanism unavailable in heterotic models. This new ingredient automatically addresses a number of puzzles present in traditional GUT models. The internal U(1) hyperflux allows us to solve the doublet-triplet splitting problem, and explains the qualitative features of the distorted GUT mass relations for lighter generations due to the Aharanov-Bohm effect. These models typically come with nearly exact global symmetries which prevent bare μ terms and also forbid dangerous baryon number violating operators. Strong curvature around our brane leads to a repulsion mechanism for Landau wave functions for neutral fields. This leads to large hierarchies of the form $\exp(-c/\varepsilon^{2\gamma})$ where c and γ are order one parameters and $\varepsilon \sim \alpha_{\mathrm{GUT}}^{-1} M_{\mathrm{GUT}}/M_{\mathrm{pl}}$. This effect can simultaneously generate a viably small μ term as well as an acceptable Dirac neutrino mass on the order of $0.5 \times 10^{-2\pm 0.5}$ eV. In another scenario, we find a modified seesaw mechanism which predicts that the light neutrinos have masses in the expected range while the Majorana mass term for the heavy neutrinos is $\sim 3 \times 10^{12\pm 1.5}$ GeV. Communicating supersymmetry breaking to the MSSM can be elegantly realized through gauge mediation. In one scenario, the same repulsion mechanism also leads to messenger masses which are naturally much lighter than the GUT scale.

Keywords: Brane Dynamics in Gauge Theories, F-Theory, Intersecting Branes Models, Superstring Vacua.

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Contents

1. Introduction

Despite many theoretical advances in our understanding of string theory, this progress has not produced a single verifiable prediction which can be tested against available experiments. Part of the problem is that in its current formulation, string theory admits a vast landscape of consistent low energy vacua which look more or less like the real world.

Reinforcing this gloomy state of affairs is the fact that the particle content of the Standard Model is generically of the type encountered in string theory. Indeed, the gauge group of the Standard Model is of the form $\prod_i \mathrm{U}(N_i)$ and the chiral matter content corresponds to bi-fundamental fields transforming in representations such as $(N_i, \overline{N}_j)$. While this may reinforce the idea that string theory is on the right track, precisely because this appears to be such a generic feature of string constructions, this also unfortunately limits the predictivity of the theory. To rectify this situation, we must impose additional criteria to narrow down the search in the vast landscape.

From a top down approach, one idea is to further incorporate some specifically stringy principles. For instance, we have learned that the large N limit of many $\mathrm{U}(N)$ gauge theories causes the gauge system to 'melt' into a dual gravitational background [1]. Moreover, this large N gauge theory can undergo a duality cascade to a small N gauge theory [2]. Indeed, the Standard Model could potentially emerge at the end of such a process. In the string theory literature, this idea has been explored in [3 − 5]. Interesting as this idea is, it does not incorporate the idea of grand unification of the gauge forces into one gauge factor in any way.

From a bottom up approach, it is natural to ask whether there is some way to incorporate the important fact that the gauge coupling constants of:

$$G_{\mathrm{std}} \equiv \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{1.1}$$

seem to unify in the minimal supersymmetric extension of the Standard Model (MSSM). This not only supports the idea that supersymmetry is realized at low energies, but also suggests that the multiple gauge group factors of the Standard Model unify into a single simple group such as SU(5) or SO(10). Moreover, the fact that the matter content of the Standard Model economically organizes into representations of the groups SU(5) and SO(10) provides a strong hint that the basic idea of grand unified theories (GUTs) is correct. For example, it is quite intriguing that all of the chiral matter of a single generation *precisely* organizes into the spinor representation 16 of SO(10). Hence, we ask whether the principle of grand unification can narrow down the large list of candidate vacua in the landscape to a more tractable, and predictive subset.

Despite the many attractive features of the basic GUT framework, the simplest implementations of this idea in four-dimensional models suffer from some serious drawbacks. For example, the minimal four-dimensional supersymmetric SU(5) GUT with standard Higgs content seems to be inconsistent with present bounds on proton decay [6]. In the absence of higher dimensional representations of SU(5) or somewhat elaborate higher dimension operator contributions to the effective superpotential, this model also leads to mass relations and over-simplified mixing matrices which are generically too strong to be correct.

This presents an opportunity for string theory to intervene: Can string theory preserve the nice features of GUT models while avoiding their drawbacks?

Indeed, the $E_8 \times E_8$ heterotic string seems very successful in this regard because the usual GUT groups SU(5) and SO(10) can naturally embed in one of the E_8 factors. See [7] for an early review on how GUT models could potentially originate from compactifying the heterotic string on a Calabi-Yau threefold. Moreover, because no appropriate four-dimensional GUT Higgs field is typically available to break the GUT group to the Standard Model gauge group, it is necessary to employ a higher-dimensional breaking mechanism. When the internal space has non-trivial fundamental group, the gauge group can break via a discrete Wilson line. In this way, the gauge group in four dimensions is always the Standard Model gauge group but the matter content and gauge couplings still unify. Moreover, such higher dimensional GUTs provide natural mechanisms to suppress proton decay and avoid unwanted mass relations. See [8 − 12] for some recent attempts in this direction.

However, the heterotic string has its own drawbacks simply because it is rather difficult to break the gauge symmetry down to G_{std}.[1] One popular method is to use internal Wilson lines to directly break the gauge symmetry to that of the MSSM. This requires that the fundamental group of the Calabi-Yau must be non-trivial. Although this can certainly be arranged, the generic Calabi-Yau threefold is simply connected and this mechanism is unavailable. Moreover, when the GUT group has rank five or higher, gauge group breaking by Wilson lines can also leave behind additional massless U(1) gauge bosons besides U(1) hypercharge. Present constraints on additional long rang forces are quite stringent, and in many cases it is not always clear how to remove these unwanted states from the low energy spectrum. In the absence of a basic principle which naturally favors a non-trivial fundamental group, it therefore seems reasonable to look for other potential realizations of the GUT paradigm in string theory.

There are two other natural ways that GUTs can appear in string theory. These possibilities correspond to non-perturbatively realized four-dimensional $\mathcal{N} = 1$ compactifications of type IIA and IIB string theory. In the type IIA case, the GUT models originate from the compactification of M-theory on manifolds with G_2 holonomy. For type IIB theories, the corresponding vacua are realized as compactifications of F-theory on Calabi-Yau fourfolds. In the latter case, the gauge theory degrees of freedom of the GUT localize on the worldvolume of a non-perturbative seven-brane. The ADE gauge group on the seven-brane corresponds to the discriminant locus of the elliptic model where the degeneration is locally of ADE type. Of these two possibilities, the holomorphic geometry of Calabi-Yau manifolds provides a more tractable starting point for addressing detailed model building issues. It was with this aim that we initiated an analysis of how GUT models can be realized in F-theory [15]. See [16, 17] for related discussions in the context of F-theory/heterotic duality.

Even so, there is a certain tension between string theory and the GUT paradigm. From a top down perspective, it is a priori unclear why there should be any distinction between

[1]At a pragmatic level, the perturbative regime of the heterotic string also seems to be inconsistent with the relation between the GUT scale M_{GUT} and the four-dimensional Planck scale M_{pl}. A discussion of this discrepancy and related issues may be found in [13]. One potential way to bypass this problem requires going to the regime of strong coupling [14].

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the Planck scale $M_{\rm pl}$ and the GUT scale $M_{\rm GUT}$. In the bottom up approach, the situation is completely reversed. Indeed, insofar as effective field theory is valid at the GUT scale, it is quite important that $M_{\rm GUT}/M_{\rm pl}$ is small and *not* an order one number. For example, in the extreme situation where the only chiral matter content of a four-dimensional GUT model originates from the MSSM, the resulting theory is asymptotically free.

In geometrically engineered gauge theories in string theory, asymptotic freedom translates to the existence of a consistent decompactification limit. It is therefore quite natural to ask if *at least in principle* we could have decoupled the two scales $M_{\rm GUT}$ and $M_{\rm pl}$. This is also in accord with the bottom up approach to string phenomenology [18 − 21] . In the present paper our main focus will therefore be to search for vacua which at least in principle admit a limit where $M_{\rm pl} \to \infty$ while $M_{\rm GUT}$ remains finite. Of course, in realistic applications $M_{\rm pl}$ should also remain finite. For completeness, we shall also present some examples of models where $M_{\rm GUT}$ and $M_{\rm pl}$ cannot be decoupled. In such cases, we note that it is not a priori clear whether the correct value of $M_{\rm GUT}$ can be achieved.

Nevertheless, the mere *existence* of a decoupling limit turns out to endow the resulting candidate models with surprising predictive power. It turns out that the only way to achieve such a decoupling limit requires that the spacetime filling seven-brane must wrap a del Pezzo surface. The fact that the relevant part of the internal geometry in this setup is limited to just ten distinct topological types is very welcome! In a certain sense, there is a unique choice corresponding to the del Pezzo 8 surface because all of the other del Pezzo surfaces can be obtained from this one by blowing down various two-cycles.

At the next level of analysis, we must determine what kind of seven-brane should wrap the del Pezzo surface. As explained in [15], realizing the primary ingredients of GUT models requires that the singularity type associated with the seven-brane should correspond to a subgroup of the exceptional group E_8. Because the Standard Model gauge group has rank four, this determines a lower bound on the rank of any putative GUT group. At rank four, SU(5) is the only available GUT group. Hence, the most 'minimal' choice is to have an SU(5) seven-brane wrapping the del Pezzo 8 surface. We will indeed find that this minimal scenario is viable. The upper bound on the rank of a candidate GUT group is six. This bound comes about from the fact that if the rank is any higher, the model will generically contain localized light degrees of freedom at points on the del Pezzo surface which do not appear to admit a standard interpretation in gauge theory [22, 15]. This is because on complex codimension one subspaces, the rank of the gauge group goes up by one, and on complex codimension two subspaces, i.e. points, the rank goes up by two. Hence, if the rank is greater than six, the compactification contains points on the del Pezzo with singularities of rank nine and higher which do not admit a standard gauge theoretic interpretation because E_8 is the maximal compact exceptional group.

In the minimal scenario where the seven-brane has gauge group SU(5), we find that there is an essentially unique mechanism by which the GUT group can break to a four-dimensional model with gauge group $G_{\rm std}$. This breaking pattern occurs in vacua where the U(1) hypercharge flux in the internal directions of the seven-brane is non-trivial. This mechanism is unavailable in heterotic compactifications because the U(1) hypercharge always develops a string scale mass via the Green-Schwarz mechanism [23]. As noted for example in [23], in order to preserve a massless U(1) hypercharge gauge boson, additional U(1) factors must mix non-trivially with this direction, which runs somewhat counter to the idea of grand unification. Nevertheless, for suitable values of the gauge coupling constants for these other factors, a semblance of unification can be maintained. See [24 – 27] for further discussion on vacua of this type.

In F-theory, we show that there is no such generic obstruction. This is a consequence of the fact that while the cohomology class of the flux on the seven-brane can be non-trivial, it can nevertheless represent a trivial class in the base of the F-theory compactification. This topological condition is necessary and also sufficient for the corresponding four-dimensional U(1) gauge boson to remain massless. An important consequence of this fact is that these F-theory vacua do not possess a heterotic dual.

The particular choice of internal U(1) flux which breaks the GUT group is also unique. To see how this comes about, we first recall that the middle cohomology of the del Pezzo 8 surface splits as the span of the canonical class and the collection of two-cycles orthogonal to this one-dimensional lattice. With respect to the intersection form on two-cycles, this orthogonal subspace corresponds to the root lattice of E_8. Moreover, the admissible fluxes of the U(1) hypercharge are in one to one correspondence with the roots of E_8. This restriction occurs because for more generic choices of U(1) flux, the low energy spectrum contains exotic matter which if present would ruin the unification of the gauge coupling constants. In keeping with the general philosophy outlined in [15], we always specify the appropriate line bundle first and only then determine whether an appropriate Kähler class exists so that the vacuum is supersymmetric. In this sense, there is a unique choice of flux because the Weyl group of E_8 acts transitively on the roots of E_8. On general grounds, this internal flux will also induce a small threshold correction near the GUT scale. Determining the size and sign of this correction would clearly be of interest to study.[2]

The matter and Higgs fields localize on Riemann surfaces in the del Pezzo surface. In F-theory, these Riemann surfaces are located at the intersection between the GUT model seven-brane and additional seven-branes in the full compactification. Along these intersections, the rank of the singularity type increases by one. This severely limits the available representation content so that the matter fields can only transform in the 5 or $\overline{5}$ along an enhancement to SU(6) and the 10 or $\overline{10}$ for local enhancement to SO(10).

The internal hypercharge flux automatically distinguishes the Higgs fields from the other chiral matter content of the MSSM. The Higgs fields localize on matter curves where the U(1) hypercharge flux is non-vanishing, and the chiral matter of the MSSM localizes on Riemann surfaces where the net flux vanishes. In other words, the two-cycles for the Higgs curves intersect the root corresponding to this internal flux while all the other chiral matter of the MSSM localizes on two-cycles orthogonal to this choice of flux. This internal choice of flux implies that the chiral matter content will always fill out complete representations of SU(5), while the Higgs doublets can never complete to full GUT multiplets. Moreover, by a suitable choice of flux on the other seven-branes, the spectrum will contain no extraneous Higgs triplets, thus solving the doublet-triplet splitting problem. In certain cases,

[2]After our work appeared, this question has been studied in [28, 29].

superheavy Higgs triplets can still cause the proton to decay too quickly. In traditional four-dimensional GUT models the missing partner mechanism is often invoked to avoid generating dangerous dimension five operators which violate baryon number. Here, this condition translates into the simple geometric condition that the Higgs up and down fields must localize on distinct matter curves.

In our study of Yukawa couplings, we shall occasionally encounter situations involving two fields charged under the GUT group and one neutral field (for example a $1 \times 5 \times \overline{5}$ interaction). In such cases, the neutral field lives on a matter curve normal to the del Pezzo which intersects this surface at a point. In order to determine the strength of the Yukawa couplings, we need to estimate the strength of the corresponding zero mode wave functions at the intersection point. It turns out that since the del Pezzo is strongly positively curved ($\mathcal{R} \sim M_{\mathrm{GUT}}^2$), the normal geometry is negatively curved. Moreover, this leads to the wave function being either attracted to, or repelled away from our brane, depending on the choice of the gauge flux on the normal intersecting seven-branes. In one case the wave function is attracted to our seven-brane, making it behave as if the wave function is localized inside the brane. In another case the wave function is repelled away from our brane, leading to an exponentially small amplitude at our brane. The exponential hierarchy is given by $\exp(-cR_\perp^2/R_{\mathrm{GUT}}^2)$ where c is a positive order one constant, $R_\perp$ is the radius of the normal geometry to the brane, and R_{GUT} is the length associated to GUT. The estimate for $R_\perp$ depends on assumptions about how the geometry normal to our brane looks, and in particular to what extent it is tubular. We find that:

$$\frac{R_\perp}{R_{\mathrm{GUT}}} = \varepsilon^{-\gamma} \tag{1.2}$$

where $1/3 \lesssim \gamma \lesssim 1$ is a measure of the normal eccentricity and ε is a small parameter:

$$\varepsilon \sim \frac{M_{\mathrm{GUT}}}{\alpha_{\mathrm{GUT}} M_{\mathrm{pl}}} \sim 7.5 \times 10^{-2}. \tag{1.3}$$

This leads to a natural hierarchy given by

$$\exp\left(-c\frac{R_\perp^2}{R_{\mathrm{GUT}}^2}\right) \sim \exp\left(-c\frac{1}{\varepsilon^{2\gamma}}\right). \tag{1.4}$$

There are various vector-like pairs which can only develop a mass through a cubic Yukawa coupling with a third field coming from a neutral normal wave function. This suppression mechanism will be useful in many such cases, including solving the μ problem and also obtaining a small Dirac neutrino mass leading to realistic light neutrino masses without using the seesaw mechanism.

There are two ways we can solve the μ problem. Perhaps most simply, we can consider geometries where the Higgs up and down fields localize on distinct matter curves which do not intersect. In this case, the μ term is identically zero. When these curves do intersect, the value of the μ term depends on the details of a gauge singlet wave function which localizes on a matter curve normal to the del Pezzo surface. In the case of attraction, the

μ term is near the GUT scale, which is untenable. In the repulsive case, the μ term is suppressed to a much lower value:

$$\frac{\mu}{M_{\mathrm{GUT}}} \sim \exp\left(-c\frac{1}{\varepsilon^{2\gamma}}\right), \tag{1.5}$$

so that the resulting value of μ can then naturally fall in a phenomenologically viable range.

In fact, a similar exponential suppression in the wave functions of the right-handed neutrinos can generate small Dirac neutrino masses of the form:

$$m_\nu^D \sim \frac{\mu\varepsilon^{-\gamma}}{\langle H_u\rangle} \times \frac{\langle H_u\rangle^2}{M_{\mathrm{GUT}}} \sim 0.5 \times 10^{-2\pm 0.5}\ \mathrm{eV} \tag{1.6}$$

which differs by a factor of $\mu\varepsilon^{-\gamma}/\langle H_u\rangle$ from the value predicted by the simplest type of seesaw mechanisms with Majorana masses at the GUT scale. We note that the value we obtain is in reasonable agreement with recent experimental results on neutrino oscillations. In this case, the Majorana mass term must identically vanish to remain in accord with observation.

A variant of the standard seesaw mechanism is also available when the right-handed neutrino wave functions are attracted to the del Pezzo surface. In this case, the Majorana mass terms in the neutrino sector are suppressed by some overall volume factors. Although the standard seesaw mechanism again generates naturally light neutrino masses $\sim 2 \times 10^{-1\pm 1.5}$ eV, we find that the Majorana mass term is naturally somewhat lighter than the GUT scale and is on the order of $\sim 3 \times 10^{12\pm 1.5}$ GeV. It is interesting that the numerical values we obtain in either scenario are both in a range of values consistent with leptogenesis, as well as the observed light neutrino masses.

Non-trivial flavor structures can potentially arise in a number of ways in this class of models. For example, one common approach in the model building literature is to use a discrete symmetry to induce additional structure in the form of the Yukawa couplings. The Weyl group symmetries of the exceptional groups naturally act on the del Pezzo surfaces. This symmetry can be partially broken by the choice of the Kähler classes of two-cycles. This may potentially lead to a model of flavor based on the discrete symmetry groups S_3, A_4 or S_4. Indeed, these are all subgroups of the Weyl group of E_8.

One of the main conceptual issues with the usual GUT framework is to explain why $m_b \sim m_\tau$ at the GUT scale while the lighter generations do not satisfy such a simple mass relation. At a qualitative level, the behavior of the omnipresent internal U(1) hypercharge flux again plays a central role in the resolution of this issue. Although the net hypercharge flux vanishes on curves which support full GUT multiplets, in general it will not vanish pointwise. Hence, the hypercharge flux can still leave behind an important imprint on the wave functions of the fields in the MSSM. Indeed, because the individual components of a GUT multiplet have different hypercharge, the Aharonov-Bohm effect will alter the distinct components of a GUT multiplet differently, leading to violations in the most naive mass relations. In fact, because the mass of a generation is higher the smaller the volume of the matter curve, the amount of flux which can pierce the curve also decreases. In this way, the most naive mass relations remain approximately intact for the heaviest generation but will in general receive corrections for the lighter generations.

In the next to minimal GUT scenario, we can consider seven-branes where the bulk gauge group has rank five. In this case there are three choices corresponding to SO(10), SO(11) and SU(6). In this paper we mainly focus on the SO(10) $= E_5$ case because it fits most closely with our general philosophy that the exceptional groups play a distinguished role in GUT models. It turns out that this model can only descend to the MSSM by a sequence of breaking patterns where the eight-dimensional theory first breaks to a four-dimensional flipped SU(5) model with gauge group SU(5)$\times$U(1). The model then operates as a traditional four-dimensional flipped SU(5) GUT which breaks to the Standard Model gauge group when a field in the 10_{-1} of SU(5) $\times$ U(1) develops a suitable vev. Indeed, direct breaking of SO(10) to the Standard Model gauge group via fluxes taking values in a U(1) $\times$ U(1) subgroup always generates exotic matter which would ruin the unification of the gauge coupling constants. Many of the more refined features of these models such as textures and our solution to the μ problem share a common origin to those studied in the minimal SU(5) model.

Even though our main emphasis in this paper is on models which admit a decoupling limit, we also consider models where such a limit does not exist. In such cases the problem of engineering a GUT model becomes more flexible because the local model is incomplete. We study examples of this situation because there are well-known difficulties in heterotic models in realizing traditional four-dimensional GUT group breaking via fields in the adjoint representation. This is due to the fact that in many cases, the requisite adjoint-valued fields do not exist. Indeed, gauge group breaking by Wilson lines is not so much an elegant ingredient in heterotic constructions as much as it is a necessary element of any construction.[3] Gauge group breaking via Wilson lines can also occur in F-theory when the surface wrapped by the seven-brane has non-trivial fundamental group. For example, a well-studied surface with $\pi_1(S) \neq 0$ is the Enriques surface which can be viewed as the $\mathbb{Z}_2$ quotient of a $K3$ surface.

Given the large proliferation of four-dimensional GUT models which exist in the model building literature, it is also natural to ask whether there exist purely four-dimensional GUT models in F-theory with adjoint-valued GUT Higgs fields. We find that this can be done provided the surface wrapped by the seven-brane has non-zero Hodge number $h^{2,0}(S) \neq 0$. But in contrast to the usual approach to four-dimensional effective field theories where it is common to assume that Planck scale physics can in principle be decoupled, here we see that the traditional four-dimensional GUT *cannot* be decoupled from Planck scale physics.

We also briefly consider supersymmetry breaking in our setup. This is surprisingly simple to accommodate because extra messenger fields can naturally arise from additional matter curves which do not intersect any of the other curves on which the matter content of the MSSM localizes. Supersymmetry breaking can then communicate to the MSSM via the usual gauge mediation mechanism. We note that because the μ term naturally devel-

[3]It is also possible to avoid this constraint in heterotic models which descend to a four-dimensional flipped SU(5) GUT. See [30, 31, 25] for further details on this approach. We also note that in certain cases, chiral superfields transforming in other representations can arise from higher Kac-Moody levels of the heterotic string.

ops a value around the electroweak scale independently of any supersymmetry breaking mechanism, we can retain many of the best features of gauge mediation such as the absence of additional flavor changing neutral currents (FCNCs) while avoiding some of the problematic elements of this scenario which are related to generating appropriate values for the μ and $B\mu$ terms. Depending on the local behavior of the wave functions which propagate in directions normal to the del Pezzo surface, the messenger scale can quite flexibly range from values slightly below the GUT scale to much lower but still phenomenologically viable mass scales.

The organization of this paper is as follows. In section 2, we formulate what we wish to achieve in our GUT constructions. In section 3 we review and slightly extend our previous work on realizing GUT models in F-theory. To this end, we describe many of the necessary ingredients for an analysis of the matter content and interaction terms of any potential model. Before proceeding to any particular class of models, in section 4 we discuss the various mass scales which will generically appear throughout this paper. In section 5, we give a general overview of the class of GUT models in F-theory we shall study. These models intrinsically divide based on how the GUT breaks to the MSSM. We first study models where the GUT scale cannot be decoupled from the Planck scale. In section 6 we discuss models where GUT breaking proceeds just as in four-dimensional models. Next, we discuss GUT breaking via discrete Wilson lines in section 7. In the remainder of the paper we focus on the primary case of interest where a decoupling limit exists. Section 8 reviews some relevant geometrical facts about del Pezzo surfaces. This is followed in section 9 by a study of GUT breaking to the MSSM via an internal U(1) hypercharge flux. In section 10 we determine which bulk gauge groups can break directly to the Standard Model gauge group via internal fluxes. We also explain in greater detail how to obtain the exact spectrum of the MSSM from such models. In section 11 we discuss a geometric realization of matter parity, and in section 12 we study the interrelation between proton decay and doublet triplet splitting in our models. After giving a simple criterion for avoiding the simplest dimension five operators responsible for proton decay, in section 13 we explain how extra global U(1) symmetries in the low energy effective theory are encoded geometrically in F-theory, and in particular, how these symmetries can forbid potentially dangerous higher dimension operators. In section 14 we discuss some coarse properties of Yukawa couplings and also speculate on how further details of flavor physics could in principle be incorporated. In this same section we also provide a qualitative explanation for why the usual mass relations of GUT models become increasingly distorted as the mass of a generation decreases. In section 15 we show that interaction terms involving matter fields which localize on Riemann surfaces outside of the surface can generate hierarchically small values for both the μ term as well as Dirac neutrino masses. We also study a variant on the usual seesaw mechanism which generates the expected mass scale for the light neutrinos. Intriguingly, the Majorana mass of the right-handed neutrinos is somewhat lower than the value expected in typical GUT models. In section 16 we propose how supersymmetry breaking could communicate to the MSSM, and in section 17 we present an SU(5) model which incorporates some (but not all!) of the mechanisms developed in previous sections. Our expectation is that further refinements are possible which are potentially more realistic. In a similar vein, in section 18

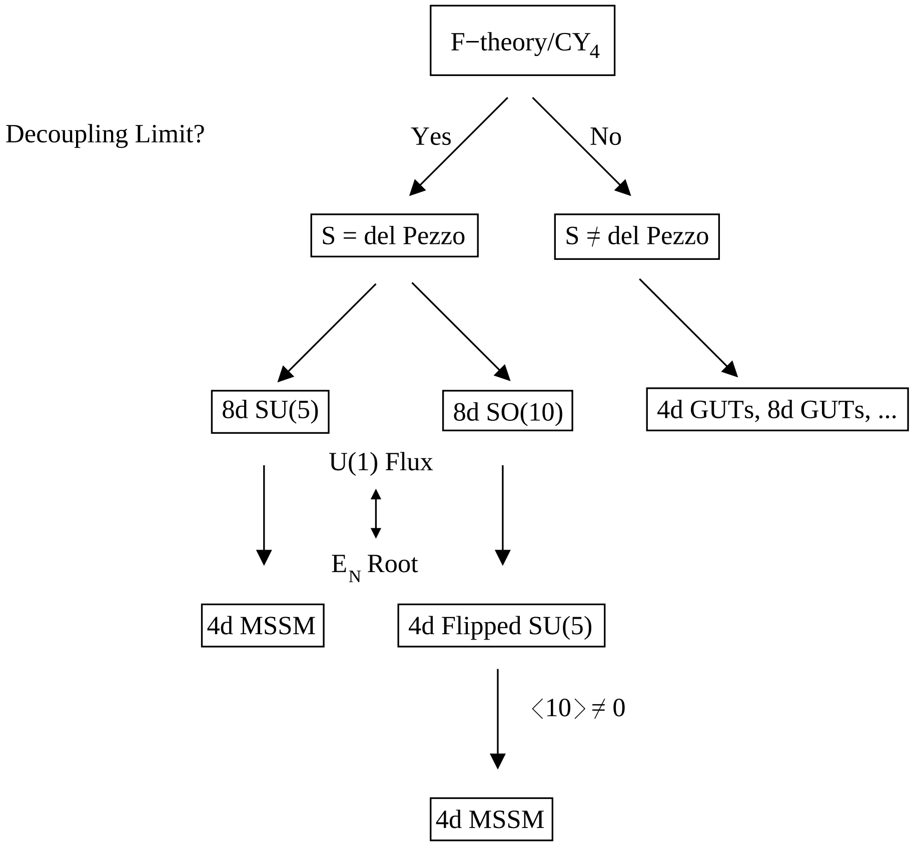

Figure 1: General overview of how GUT breaking constrains the type of GUT model.

we present a flipped SU(5) model. Section 19 collects various numerical estimates obtained throughout the paper, and section 20 presents our conclusions. The appendices contain further background material used in the main body of the paper and which may also be of use in future model building efforts.

2. Constraints from low energy physics

In this section we define the criteria by which we shall evaluate how successfully our models reproduce features of low energy physics obtained by a minimal extrapolation of experimental data to the MSSM. There are a number of open questions in both phenomenology and string theory which must ultimately be addressed in any approach. See [32, 33] for an expanded discussion of some of the issues we briefly address here.

At the crudest level, we require that any viable model contain precisely three generations of chiral matter. It is an experimental fact that the chiral matter content of the Standard Model organizes into SU(5) and SO(10) GUT multiplets. Coupled with the fact that the gauge couplings of the MSSM appear to unify at an energy scale $M_{\text{GUT}} \sim 3 \times 10^{16}$ GeV, we shall aim to reproduce these features in all of the models we shall consider. For all of these reasons, we require that the low energy content of all of our models must match to the matter content of the MSSM. By this we mean that in addition to achieving the correct chiral matter content and Higgs content of the MSSM, all additional matter charged under the gauge groups must at the very least fit into vector-like pairs of complete GUT multiplets in order to retain gauge coupling unification.[4] In the minimal incarnation of GUT models considered here, we shall further require that the low energy spectrum of particles charged under the Standard Model gauge group must exactly match to the matter content of the MSSM. We note that historically, even this qualitative requirement has been difficult to achieve in Calabi-Yau compactifications of the perturbative heterotic string.

Although the correct particle content is a necessary step in achieving a realistic model, it is certainly not sufficient because we must also reproduce the superpotential of the MSSM:

$$W = \mu H_u H_d + \lambda^u_{ij} Q^i U^j H_u + \lambda^d_{ij} Q^i D^j H_d + \lambda^l_{ij} L^i E^j H_d + \lambda^\nu_{ij} L^i N^j_R H_u + \dots \tag{2.1}$$

where the indices i and j label the three generations. While the precise form of the Yukawa matrices labeled by the λ's will lead to masses and mixing terms between the generations, a necessary first step is that there are in principle non-zero contributions to the above superpotential! As a first approximation, we require that the tree level superpotential of the theory at high energy scales generate a non-trivial interaction term for the third generation so that there is a rough hierarchy in mass scales. In the context of GUT models, it is well-known that because the particle content of the Standard Model organizes into complete GUT multiplets, the Yukawa couplings couple universally to fields organized in such multiplets. One attractive feature of the tree level superpotential in most GUT models is that the third generation obeys a simple mass relation of the form $m_b/m_\tau \sim 1$ at the GUT scale. Evolving this relation under the renormalization group to the weak scale yields the relation $m_b/m_\tau \sim 3$ which is roughly in agreement with experiment. Unfortunately, this relation is violated for the lighter generations. Ideally, it would be of interest to find models which naturally preserve the mass relations of the third generation while modifying the relations of the first two generations.

At the next level of approximation, any model should be consistent with current experimental bounds on the lifetime of the proton ($\geq 10^{31} - 10^{33}$ yrs [36]). This requires that certain operators must be absent or sufficiently suppressed in the low energy superpotential. Indeed, note that in equation (2.1), we have implicitly only included renormalizable R-parity invariant couplings because if present, the interaction terms $\lambda_{ijk} U^i D^j D^k$ and $\lambda'_{ijk} L^i L^j E^k$ will cause the proton to decay too rapidly. We shall consider models with and without R-parity. In the latter case, we therefore must present alternative reasons to expect renormalizable operators responsible for R-parity to vanish.

Proton decay is a hallmark of GUT models. Aside from renormalizable interaction terms, the dominant contribution to proton decay in the simplest GUT models comes from

[4] While it is in principle possible to consider models where vector-like exotics preserve gauge coupling unification, we believe this runs contrary to the spirit of GUT models. Although we shall not entertain this possibility here, see [34, 35] for further discussion of this possibility.

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the dimension five operator [37, 38]:[5]

$$O_5 = \frac{c_5}{M_{\rm GUT}} \int d^2\theta QQQL \tag{2.2}$$

and the dimension six operator:

$$O_6 = \frac{c_6}{M_{\rm GUT}^2} \int d^4\theta QQU^\dagger E^\dagger . \tag{2.3}$$

The operator O_5 can originate from the exchange of heavy Higgs triplets and can cause the decay $p \to K^+\overline{\nu}$. The operator O_6 can originate from the exchange of heavy off-diagonal GUT group gauge bosons and can cause the decay $p \to e^+\pi_0$. To remain in accord with current bounds on nucleon decay, c_6 can typically be an order one coefficient whereas c_5 must be suppressed at least to the order of 10^{-7}. See [39] for further discussion on proton decay in GUT models.

In four-dimensional GUT models, this issue is closely related to the mechanism responsible for removing the Higgs triplets from the low energy spectrum. One common approach is to invoke some continuous or discrete symmetry to sufficiently suppress this operator. The use of discrete symmetries in compactifications of M-theory on manifolds with G_2 holonomy has been studied in [40]. Note that while the Higgs triplet must develop a sufficiently large mass in order to reproduce the particle content of the MSSM, we must also require that the supersymmetric Higgs mass μ should be on the order of the weak scale.

While the above problems are necessary requirements for any potentially viable model, there are many additional phenomenological constraints which must be satisfied in a fully realistic compactification. In principle, a complete model should also naturally accommodate hierarchical masses for the quarks and leptons. For example, in conventional GUT models, the seesaw mechanism allows the neutrino masses in the Standard Model to be much lighter than the electroweak scale. At a more refined level, a full model should explain why the CKM matrix is nearly equal to the identity matrix whereas the MNS matrix contains nearly maximal mixing between the neutrinos.

A fully realistic model must of course specify how supersymmetry is broken and provide a mechanism for communicating this breaking to the MSSM. Our expectation is that this issue can be treated independently from the supersymmetric models which shall be our primary focus here. We note that for general string compactifications, supersymmetry breaking is closely entangled with moduli stabilization. While we will not specify a method for stabilizing moduli, we note that F-theory provides a natural arena for further study of this issue. See [41] for a particular example of moduli stabilization in F-theory and [42] for a review of this active area of research.

3. Basic setup

In this section we review the basic properties of exceptional seven-branes in F-theory. In particular, we explain how to compute the low energy matter spectrum as well as the effective superpotential of the four-dimensional theory. Further details may be found in [15].

[5]There is an additional contribution to the superpotential given by $UUDE$. At the level of discussion in this paper, it is sufficient to only deal with the term $QQQL$.

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F-theory compactified on an elliptically fibered Calabi-Yau fourfold preserves $\mathcal{N}=1$ supersymmetry in the four uncompactified spacetime dimensions. Letting B_3 denote the base of the Calabi-Yau fourfold, the discriminant locus of the elliptic fibration determines a subvariety Δ of complex codimension one in the base B_3. Denoting by S the Kähler surface defined by an irreducible component of Δ, when this degeneration locus is a singularity of ADE type, the resulting eight-dimensional theory defines the worldvolume of an exceptional seven-brane with gauge group G_S of ADE type. This singularity type can enhance along complex codimension one curves in S to a singularity of type G_Σ and can further enhance at complex codimension two points in S to a singularity of type G_p. Such points correspond to the triple intersection of three matter curves. Because the Cartan subalgebra of each singularity type is visible to the geometry [43, 44], these enhancements satisfy the containment relations:

$$G_S \times \mathrm{U}(1) \times \mathrm{U}(1) \subset G_\Sigma \times \mathrm{U}(1) \subset G_p. \tag{3.1}$$

As argued in [15], many necessary features of even semi-realistic GUT models require that $G_p \subset E_8$. In particular, this implies that the rank of the bulk gauge group G_S is at most six. This significantly limits the available bulk gauge groups because the rank of G_S must be at least four in order to contain the Standard Model gauge group.

In this paper we shall assume that given a choice of matter curves, there exists a Calabi-Yau fourfold which contains the corresponding local enhancement in singularity type. While this assumption is clearly not fully justified for compact models, in the context of local models this can always be done. As an example, we now engineer a local model where the bulk gauge group E_6 enhances along a matter curve Σ in S to an E_7 singularity. A local elliptic model of this type is:

$$y^2 = x^3 + fxz^3 + q^2 z^4. \tag{3.2}$$

In the above, q is a section of $\mathcal{O}_S(\Sigma)$, f is a section of $L \otimes K_S^{-3}$ and the coordinates (x, y, z) transform as a section of [15]:

$$L^2 \oplus L^3 \oplus L \otimes K_S \tag{3.3}$$

where K_S denotes the canonical bundle on S and L is a line bundle which can be expressed in terms of K_S and $\mathcal{O}_S(\Sigma)$. The essential point of this example is that in a local model, there always exists a line bundle L such that the resulting local model is well-defined. For example, in this case we have:

$$L = \mathcal{O}_S(\Sigma) \otimes K_S^2. \tag{3.4}$$

Further, we shall make the additional assumption that there is no mathematical obstruction to various twofold enhancements in the rank of the singularity at points of the surface S. It would certainly be of interest to study this issue.

We now describe in greater detail the effective action of exceptional seven-branes. In terms of four-dimensional $\mathcal{N}=1$ superfields, the matter content of the theory consists of an $\mathcal{N}=1$ vector multiplet which transforms as a scalar on S, a collection of chiral superfields $\mathbb{A}_{\bar{i}}$ which transform as a $(0,1)$ form on S (the bulk gauge bosons) and a collection of chiral superfields Φ which transform as a holomorphic $(2,0)$ form on S. The bulk modes couple through the superpotential term:

$$W_S = \int_S Tr\left[\left(\overline{\partial}\mathbb{A} + \mathbb{A}\wedge\mathbb{A}\right)\wedge\Phi\right]. \tag{3.5}$$

When two irreducible components S and S' of Δ intersect on a Riemann surface Σ, the singularity type enhances further. In this case, additional six-dimensional hypermultiplets localize along Σ. As in [44], the representation content of these fields is given by decomposing the adjoint representation of the enhanced singularity to the product $G_S \times G_{S'}$ associated with the gauge groups on S and S'. In terms of four-dimensional $\mathcal{N}=1$ superfields, the matter content localized on a curve consists of chiral superfields Λ and Λ^c which transform as spinors on Σ. The bulk modes couple to matter fields localized on the curve via the superpotential term:

$$W_\Sigma = \int_\Sigma \left\langle \Lambda^c, (\overline{\partial} + \mathbb{A} + \mathbb{A}')\Lambda \right\rangle \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing which is independent of any metric data.

Finally, when three irreducible components of Δ intersect at a point p, the singularity type can enhance even further. Evaluating the overlap of three Λ's for three matter curves yields a further contribution to the four-dimensional effective superpotential:

$$W_p = \Lambda_1\Lambda_2\Lambda_3|_p. \tag{3.7}$$

An analysis similar to that given below equation (3.2) shows that given three matter curves which form a triple intersection, so long as the resulting interaction term is consistent with group theoretic considerations, there exists a local Calabi-Yau fourfold with the desired twofold enhancement in singularity type.

Having specified the individual contributions to the quasi-topological eight-dimensional theory, the superpotential is:

$$W[\Phi, A, \Lambda] = W_{S_1} + \ldots + W_{S_l} + W_{\Sigma_1} + \ldots + W_{\Sigma_m} + W_{p_1} + \ldots + W_{p_n} + W_{\text{flux}} + W_{np}. \tag{3.8}$$

In the above, the corresponding fields entering the above expression are to be viewed as a large collection of four-dimensional chiral superfields labeled by points of the complex surfaces S_i and the Riemann surfaces Σ_i. We have also included the contribution from the flux-induced superpotential which couples to the various $(2,0)$ forms of the seven-branes and indirectly to matter fields localized on curves. As explained in [15], the vevs for the $(2,0)$ form and fields localized on matter curves correspond to complex deformations of the Calabi-Yau fourfold. Because the flux-induced superpotential couples to the complex structure moduli of the Calabi-Yau fourfold, such terms will generically be present. In equation (3.8), we have also included the term W_{np} which denotes all non-perturbative contributions from wrapped Euclidean three-branes. These terms are proportional to $\exp(-aVol(S)) \sim \exp(-c/\alpha_{\text{GUT}})$ where c is an order one positive constant. In a GUT model where the gauge coupling constants unify perturbatively, such contributions are negligible.

The fields of the four-dimensional effective theory correspond to zero mode solutions in the presence of a background field configuration. As in [15], we shall confine our analysis of the matter spectrum to backgrounds where all fields other than the bulk gauge field are expanded about zero. In the presence of a non-trivial background gauge field configuration, the chiral matter content of the four-dimensional effective theory descends from bulk modes on S and Riemann surfaces which we denote by the generic label Σ. An instanton taking values in a subgroup H_S will break G_S to the commutant subgroup. Decomposing the adjoint representation of G_S to the maximal subgroup of the form $\Gamma_S \times H_S$, the chiral matter transforming in a representation τ of Γ_S descends from the bundle-valued cohomology groups:

$$\tau \in \; H^0_{\overline{\partial}}(S, \mathcal{T}^*)^* \oplus H^1_{\overline{\partial}}(S, \mathcal{T}) \oplus H^2_{\overline{\partial}}(S, \mathcal{T}^*)^* \tag{3.9}$$

where $\mathcal{T}$ denotes a bundle transforming in the representation T of H_S obtained by the decomposition of the adjoint representation of the associated principle G_S bundle on S. When S is a del Pezzo surface, the cohomology groups $H^0_{\overline{\partial}}$ and $H^2_{\overline{\partial}}$ vanish for supersymmetric gauge field configurations so that the number of zero modes transforming in the representation τ is given by an index:

$$n_\tau = \chi(S, \mathcal{T}) = -\left(1 + \frac{1}{2} c_1(S) \cdot c_1(\mathcal{T}) + \frac{1}{2} c_1(\mathcal{T}) \cdot c_1(\mathcal{T})\right). \tag{3.10}$$

An analogous computation holds for the zero mode content localized on a Riemann surface transforming in a representation $\nu \times \nu'$ of $H_S \times H_{S'}$:

$$\nu \times \nu' \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes \mathcal{V} \otimes \mathcal{V}') \tag{3.11}$$

so that the net number of zero modes is given by the index:

$$n_{\nu \times \nu'} - n_{\overline{\nu \times \nu'}} = \deg\left(\mathcal{V} \otimes \mathcal{V}'\right). \tag{3.12}$$

In many cases we shall compute the relevant cohomology groups in equation (3.11) by assuming a canonical choice of spin structure. As argued in [15], this can always be done when the curve corresponds to the vanishing locus of the holomorphic $(2,0)$ form in the eight-dimensional theory.

When $\pi_1(S) \neq 0$, it is also possible to consider vacua with non-trivial Wilson lines. In order to avoid complications from the reduction of additional supergravity modes, we shall always assume that $\pi_1(S)$ is a finite group. The discussion closely parallels a similar analysis in heterotic compactifications (see for example [45]). Recall that admissible Wilson lines are specified by a choice of element $\rho_S \in Hom(\pi_1(S), G_S)$. In order to maintain continuity with the discussion reviewed above, we shall require that the non-trivial portion of the discrete Wilson line takes values in the subgroup $\Gamma_S \subset G_S$ defined above. More generally, this restriction can be lifted and may allow additional possibilities for projecting out phenomenologically unviable representations from the low energy spectrum. Under these restrictions, the unbroken four-dimensional gauge group is given by the commutant subgroup of $\rho_S(\pi_1(S)) \times H_S$ in G_S.

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We now determine the zero mode content of the theory in the presence of a non-trivial discrete Wilson line. As in Calabi-Yau compactifications of the heterotic string, our strategy will be to lift all computations to a covering theory. Because $\pi_1(S)$ is finite, the universal cover of S denoted by $\widetilde{S}$ is a compact Kähler surface. Letting $p:\widetilde{S}\to S$ denote the covering map, the bundle $\mathcal{T}$ on S now lifts to a bundle $\widetilde{\mathcal{T}}=p^*(\mathcal{T})$ on $\widetilde{S}$. Under the present restrictions, the Wilson line corresponds to a flat Γ_S-bundle induced from the covering map from $\widetilde{S}$ to S. The deck transformation defined by the action of $\pi_1(S)$ on $\widetilde{S}$ also determines a group action of $\pi_1(S)$ on the cohomology groups $H^i_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}})$. Treating $H^i_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}})$ as a complex vector space, the eigenspace decomposition of $H^i_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}})$ is of the form:

$$H^i_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}})\simeq\underset{\lambda}{\oplus}\mathbb{C}_\lambda \tag{3.13}$$

in the obvious notation. The irreducible representation of Γ_S defined by τ decomposes into irreducible representations of the maximal subgroup $\Gamma\times\rho_S(\pi_1(S))\subset\Gamma_S$ as:

$$\tau\simeq\underset{i}{\oplus}\tau_i\otimes R_i. \tag{3.14}$$

The zero modes transforming in the representation τ_i are therefore specified by the ρ_S invariant subspaces:

$$\tau_i:\left[H^0_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}}^*)^*\otimes R_i\right]^{\rho_S}\oplus\left[H^1_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}}^*)^*\otimes R_i\right]^{\rho_S}\oplus\left[H^2_{\overline{\partial}}(\widetilde{S},\widetilde{\mathcal{T}}^*)^*\otimes R_i\right]^{\rho_S}. \tag{3.15}$$

Having specified the zero mode content of the theory, we can now in principle determine the full superpotential of the low energy effective theory by integrating out all Kaluza-Klein modes from equation (3.8). This is similar to the treatment of Chern-Simons gauge theory as a string theory [46]. For quiver gauge theories defined by D-brane probes of Calabi-Yau threefolds, the higher order terms of the effective superpotential are given by integrating out all higher Kaluza-Klein modes from the associated holomorphic Chern-Simons theory for B-branes [47].

In the present context, we can follow the procedure outlined in [48] to determine the full expression for the effective superpotential. This is given by a bosonic partition function with action given by the superpotential of equation (3.8). Viewing the higher-dimensional fields as a collection of four-dimensional chiral superfields labeled by points of the internal space, the effective superpotential is now given by the bosonic path integral:

$$\exp\left(-W_{\text{eff}}[\Phi_0,A_0,\Lambda_0]\right)=\int\limits_{1PI}[d\Phi][dA][d\Lambda]\exp\left(-W[\Phi+\Phi_0,A+A_0,\Lambda+\Lambda_0]\right) \tag{3.16}$$

where the zero subscript denotes the zero mode, and the path integral is over all one particle irreducible Feynman diagrams. In this expression, W_{tree} should be viewed as a bosonic action with functional dependence identical to that of equation (3.8). The complete four-dimensional effective superpotential for the zero modes is then determined by the partition function of the quasi-topological theory. We emphasize that this partition function is well-defined without any reference to metric data. A very similar procedure for extracting the superpotential by integrating out Kaluza-Klein modes in heterotic compactifications has been given in [23]. Some examples of similar computations for quiver gauge theories can be found in [49]. To conclude this section, we note that any symmetry of the full eight-dimensional theory descends to the four-dimensional effective superpotential for the zero modes. Neglecting the contribution due to non-perturbative effects in equation (3.8), the extra U(1) factors which are always present when the singularity type enhances will provide additional global symmetries in the effective theory which will typically forbid some higher dimension operators from being generated. Although non-perturbative effects can violate these symmetries, the corresponding contribution to $W_{\text{eff}}[\Phi_0, A_0, \Lambda_0]$ will typically be small enough that we may safely neglect such contributions.

These general considerations already constrain the matter content of any candidate theory. Modes propagating in the bulk of the surface S must transform in the adjoint representation of the bulk gauge group. Moreover, although matter fields can localize on a curve Σ inside of S, these fields must descend from the adjoint representation of G_Σ. For example, for SU(N) gauge group factors which do not embed in E_8, the only available local enhancements are to higher A or D type singularities. In such cases, the decomposition of the adjoint representation only contains two index representations. Similar restrictions apply for SO(N) gauge group factors which do not embed in E_8. In particular, the spinor representation never appears in such cases. In a sense, this is to be expected because these are precisely the types of configurations which can be realized within perturbative type IIB vacua.

For SO(N) $\subset E_8$ gauge groups, the available representations are the vector, spinor or adjoint representations, and for SU(N) $\subset E_8$ gauge groups, the only available representations are the one, two or three index anti-symmetric and the adjoint representations.[6] For example, when $G_S = \text{SO}(10)$, this implies that all of the matter fields transform in the $10, 16, \overline{16}$ or 45, while for $G_S = \text{SU}(5)$, the only available representations are the 5, $\overline{5}$, 10, $\overline{10}$ or 24. in the specific case of del Pezzo models, this matter content is even more constrained. Indeed, as explained in [15], the bulk zero mode content for del Pezzo models never contains chiral superfields which transform in the adjoint representation of the unbroken gauge group in four dimensions.

In fact, the type of twofold enhancement strongly determines the qualitative behavior of the associated triple intersection of matter curves. For example, the possible rank two enhancements of SU(5) are E_6, SO(12), and SU(7). In the case of E_6 and SO(12), the associated curves which form a triple intersection all live inside of S. Indeed, by group theory considerations, the matter fields localized on each curve transform in non-trivial representations of SU(5) [15]. On the other hand, this is qualitatively different from a local enhancement to SU(7). In this case, two of the curves of the triple intersection support matter in the fundamental and anti-fundamental of SU(5) and therefore live in S, while

[6] Strictly speaking there are additional possibilities if the rank of the bulk singularity enhances by more than one rank. If one allows more general breaking patterns involving higher SU(N) and SO($2N$) type enhancements, it is also possible to achieve two index symmetric representations of SU(N) theories. For example, letting A_{2N} denote the two index anti-symmetric representation of SU($2N$), A_{2N} decomposes to SU(N) $\times$ SU(N) as $A_{2N} \to A_N \otimes 1 + 1 \otimes A_N + F_N \otimes F_N$. Higgsing this to the diagonal SU(N) subgroup, we note that the product $F_N \otimes F_N$ contains two index symmetric representations. This is a rather exotic possibility and we shall therefore not consider it further in this paper.

the third curve of the intersection supports matter in the singlet representation.

More generally, we note that as opposed to a generic field theory, in F-theory, vector-like pairs of the bulk gauge group can only interact through cubic superpotential terms involving a field localized on a curve which intersects S at a point. While the vev of this gauge singlet can induce a mass term for the vector-like pair, the dynamics of this field in the threefold base B_3 is qualitatively different from fields which localize inside of S.

4. Mass scales and decoupling limits

Before proceeding to specific models, we first present a general analysis of the relevant mass scales in the local models we treat in this paper. Rather than specify one particular profile for the threefold base B_3, we consider both geometries where B_3 is roughly tubular so that it decomposes as the product of S with two non-compact directions orthogonal to S in B_3, as well as more homogeneous profiles. To parameterize our ignorance of the details of the geometry, we define the length scales:

$$R_S \equiv Vol(S)^{1/4} \tag{4.1}$$

$$R_B \equiv Vol(B_3)^{1/6} \tag{4.2}$$

as well as a cutoff length scale $R_\perp$ which measures the radius normal to S:

$$R_\perp \equiv R_B \times \left(\frac{R_B}{R_S}\right)^\nu \tag{4.3}$$

so that the exponent ν ranges from $\nu = 0$ when B_3 is homogeneous, to the value $\nu = 2$ when B_3 is the product of S with two non-compact directions. Indeed, the approximations we consider in this paper are valid in the regime $0 \lesssim \nu \lesssim 2$. Note that under the assumption $R_B > R_S$, the three length scales are related by:

$$R_\perp > R_B > R_S. \tag{4.4}$$

See figure 2 for a comparison of the local behavior of B_3 for $\nu \sim 0$ and $\nu \sim 2$. To clarify, although the directions normal to S are "non-compact" in our local model, in a globally consistent compactification of F-theory they will still be quite small, and all on the order of the GUT scale, as will be discussed below. Indeed, this is quite different from models based on large extra dimensions which can be either flat, but still compact [50], or potentially highly warped and of infinite extent [51].

Compactifying on a threefold base B_3, the ten-dimensional Einstein-Hilbert action is:

$$S_{\rm EH} \sim M_*^8 \int\limits_{\mathbb{R}^{3,1}\times B_3} R\sqrt{-g}d^{10}x \tag{4.5}$$

where M_* is a particular mass scale associated with the supergravity limit of the F-theory compactification. In perturbative type IIB string theory, the parameter M_* is given in

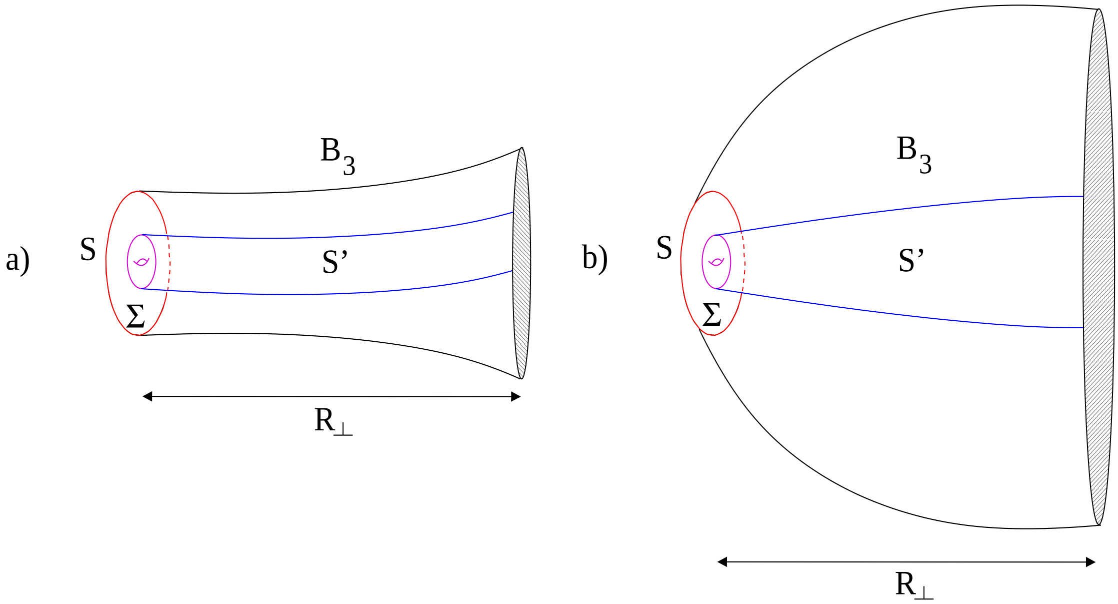

Figure 2: Depiction of F-theory compactified on a local model of a Calabi-Yau fourfold with non-compact base threefold B_3. The diagram shows the behavior of the geometry in the neighborhood of a compact Kähler surface S on which the gauge degrees of freedom of the GUT model can localize in the cases where B_3 is given by a roughly tubular geometry, as in case a), as well as geometries where B_3 is more homogeneous, as in case b). In both cases, the directions orthogonal to S in B_3 are large compared to S, but not warped. To regulate the geometry of the local model it is necessary to introduce a cutoff length scale which we denote by $R_\perp$. The intersection locus between the compact surface S and a non-compact surface S' appears as a curve Σ in S. When seven-branes wrap both surfaces, additional light states will localize on this matter curve.

string frame by the relation $M_*^8 = M_s^8/g_s^2$. Upon reduction to four dimensions, the four-dimensional Planck scale M_{pl} satisfies the relation:

$$M_{\text{pl}}^2 \sim M_*^8 Vol(B_3). \tag{4.6}$$

The tension of a seven-brane wrapping a Kähler surface S in B_3 determines the gauge coupling constant of the four-dimensional effective theory. More precisely, the coefficient of the kinetic term for the gauge field strength is of the form:[7]

$$S_{\text{kin}} \sim -M_*^4 \int\limits_{\mathbb{R}^{3,1}\times S} Tr\left(F \wedge *_8 F\right). \tag{4.7}$$

The value of the gauge coupling constant at the scale of unification is therefore:

$$\alpha_{\text{GUT}}^{-1} \sim M_*^4 Vol(S). \tag{4.8}$$

[7]The astute reader will notice a difference in sign between the gauge kinetic term used here, and the convention adopted in [15]. In [15], we adopted an anti-hermitian basis of Lie algebra generators in order to conform to conventions typically used in topological gauge theory. Because our emphasis here is on the four-dimensional effective field theory, in this paper we have reverted back to the standard sign convention in the physics literature so that all Lie group generators are hermitian.

Equations (4.6) and (4.8) now imply:

$$Vol(B_3) \sim (\alpha_{\mathrm{GUT}} M_{\mathrm{pl}} Vol(S))^2 \tag{4.9}$$

or:

$$R_B^6 \sim \left(\alpha_{\mathrm{GUT}} M_{\mathrm{pl}} R_S^4\right)^2 . \tag{4.10}$$

We now convert these geometric scales into mass scales in the low energy effective theory. To this end, we next relate $Vol(S)$ to the GUT scale M_{GUT}. In most of the cases we consider, non-zero flux in the internal directions of S will partially break the bulk gauge group of the seven-brane. Letting $\sqrt{\langle F_S \rangle}$ denote the mass scale of the internal flux, we therefore require:

$$M_{\mathrm{GUT}}^2 \sim \langle F_S \rangle . \tag{4.11}$$

Because the flux is measured in units of length^{-2} on the surface S, this implies:

$$Vol(S) \sim M_{\mathrm{GUT}}^{-4}. \tag{4.12}$$

Equation (4.9) therefore yields:

$$Vol(B_3) \sim \left(\alpha_{\mathrm{GUT}} M_{\mathrm{pl}} M_{\mathrm{GUT}}^{-4}\right)^2 . \tag{4.13}$$

The radii R_B and R_S are therefore given by:

$$\frac{1}{R_S} \sim M_{\mathrm{GUT}} = 3 \times 10^{16} \text{ GeV} \tag{4.14}$$

$$\frac{1}{R_B} \sim M_{\mathrm{GUT}} \times \varepsilon^{1/3} \sim 10^{16} \text{ GeV} \tag{4.15}$$

where we have introduced the small parameter:

$$\varepsilon \equiv \frac{M_{\mathrm{GUT}}}{\alpha_{\mathrm{GUT}} M_{\mathrm{pl}}} \sim 7.5 \times 10^{-2}. \tag{4.16}$$

Collecting equations (4.9) and (4.12), the parameter $R_\perp$ now takes the form:

$$\frac{1}{R_\perp} = M_{\mathrm{GUT}} \times \varepsilon^{\gamma} \sim 5 \times 10^{15 \pm 0.5} \text{ GeV} \tag{4.17}$$

where $1/3 \leq \gamma \leq 1$. We note that these numerical values for the radii satisfy the inequality of line (4.4).

We conclude this section by discussing the normalization of Yukawa couplings in models where the superpotential originates from the triple intersection of matter curves. In a holomorphic basis of wave functions, the F- and D-terms are:

$$L_F^{\mathrm{hol}} = \sum_p \psi_i(p)\psi_j(p)\psi_k(p) \int d^2\theta \widetilde{\phi}_i \widetilde{\phi}_j \widetilde{\phi}_k \tag{4.18}$$

$$\equiv \lambda_{ijk}^{\mathrm{hol}} \int d^2\theta \widetilde{\phi}_i \widetilde{\phi}_j \widetilde{\phi}_k \tag{4.19}$$

$$L_D^{\mathrm{hol}} = M_*^2 \int_\Sigma d^4\theta K(\widetilde{\phi}, \widetilde{\phi}^\dagger) \tag{4.20}$$

where in the above, $\psi_i(p)$ denotes the internal value of the wave function associated with the four-dimensional chiral superfield $\widetilde{\phi}_i$ evaluated at a point p in S, and the holomorphic Yukawa couplings are defined as:

$$\lambda_{ijk}^{\mathrm{hol}} = \sum_p \psi_i(p)\psi_j(p)\psi_k(p). \tag{4.21}$$

The behavior of the wave functions near these points can generate hierarchically small values near nodal points, and order one values away from such nodal points.

We eventually wish to extract numerical estimates for the physical Yukawa couplings, defined in a basis of four-dimensional chiral superfields with canonically-normalized kinetic terms. However, if we reduce the D-term in (4.20) over Σ, we find that the kinetic term for $\widetilde{\phi}$ is multiplied by the L^2-norm on Σ of the corresponding zero-mode wave function ψ.

In general, ψ transforms on Σ as a holomorphic section of $K_\Sigma^{1/2} \otimes L$, where L is a line bundle on Σ determined by the gauge field on S. Both $K_\Sigma^{1/2}$ and L carry natural hermitian metrics inherited from the bulk metric and gauge field on S. Fixing the holomorphic wave function ψ, we are interested in how the L^2-norm of ψ scales with the metric on S, since the volume of S effectively determines M_{GUT}. For concreteness, let us write the metric on S in local holomorphic coordinates (z, w) as $ds^2 = g_{z\overline{z}}\, dz d\overline{z} + g_{w\overline{w}}\, dw d\overline{w}$, where z is a local holomorphic coordinate along Σ and w is a holomorphic coordinate normal to Σ. Under an overall scaling $g \mapsto \ell g$, the hermitian metric on L is unchanged, so the norm of ψ behaves as

$$\begin{aligned} \langle\psi|\psi\rangle \;&=\; \int_\Sigma d^2z\, g_{z\overline{z}} \big(g^{z\overline{z}}\big)^{1/2} \psi\overline{\psi}\,, \\ &\longmapsto\; \ell^{1/2}\, \langle\psi|\psi\rangle\,. \end{aligned} \tag{4.22}$$

Since the volume of Σ scales with ℓ, we see from (4.22) that $\langle\psi|\psi\rangle$ scales with $\mathrm{Vol}(\Sigma)^{1/2}$.

At first glance, the dependence of $\langle\psi|\psi\rangle$ on ℓ might appear to be the only source of ℓ-dependence in the respective F- and D-terms in (4.18) and (4.20), since the F-term is determined by the overlap of fixed holomorphic wavefunctions. However, in making precise sense of this overlap, an additional ℓ-dependence also enters.

To explain this ℓ-dependence, let us consider a slightly simplified situation, for which the holomorphic curves Σ_1, Σ_2, and Σ_3 meet transversely at a point inside a Calabi-Yau threefold B_3. The role of the line bundle L is inessential, so on each curve we take the wavefunction ψ_i to transform as a holomorphic section of $K_{\Sigma_i}^{1/2}$. In local holomorphic coordinates (z, w, v) around the point p of intersection, the wavefunction overlap is defined by

$$\psi_1(p)\psi_2(p)\psi_3(p)\, \frac{\sqrt{dz}\sqrt{dw}\sqrt{dv}}{\sqrt{\Omega(p)}}\,. \tag{4.23}$$

Here Ω is a holomorphic three-form on B_3 which we must introduce so that the overlap in (4.23) does not depend on the particular holomorphic coordinates (z, w, v) chosen at p.

Of course, Ω is unique up to scale — but it is precisely the scale of the overlap that we are trying to fix! Given that B_3 carries a metric, we fix the norm of Ω by the requirement that $-i\,\Omega\wedge\overline{\Omega} = \omega\wedge\omega\wedge\omega$, where ω is the Kähler form associated to the metric on B_3. Once we impose this condition, Ω scales as $\Omega \mapsto \ell^{3/2}\,\Omega$ under an overall scaling of the metric on B_3. Hence the wavefunction overlap in (4.23) and thus the holomorphic Yukawa coupling $\lambda_{ijk}^{\rm hol}$ actually scales as $\ell^{-3/4}$.

After canonically normalizing all kinetic terms, the physical Yukawa couplings are given by

$$\lambda_{ijk}^{\rm phys} = \frac{\lambda_{ijk}^{\rm hol}}{\sqrt{M_*^2\langle\psi_i|\psi_i\rangle\, M_*^2\langle\psi_j|\psi_j\rangle\, M_*^2\langle\psi_k|\psi_k\rangle}}\,. \tag{4.24}$$

By the preceding discussion, under an overall scaling $g \mapsto \ell\, g$ of the metric on B_3, the physical Yukawa coupling scales as $\lambda_{ijk}^{\rm phys} \mapsto \ell^{-3/2}\,\lambda_{ijk}^{\rm phys}$. Restoring the dependence on the volumes of each curve, we find the result which one would naively guess,

$$\lambda_{ijk}^{\rm phys} = \frac{\lambda_{ijk}^{0}}{\sqrt{M_*^2{\rm Vol}(\Sigma_i)\, M_*^2{\rm Vol}(\Sigma_j)\, M_*^2{\rm Vol}(\Sigma_k)}}\,. \tag{4.25}$$

Here λ_{ijk}^0 denotes the fiducial, order one Yukawa coupling defined by (4.23) when B_3 has unit volume.

Although we have phrased the preceding discussion in the very special case that Σ_1, Σ_2, and Σ_3 are holomorphic curves intersecting transversely in a Calabi-Yau threefold, the result (4.25) holds quite generally in F-theory. According to the discussion in §5.2 of [15], when Σ_1, Σ_2, and Σ_3 are matter curves intersecting at a point p inside S, one must choose a trivialization of $\left(K_{\Sigma_1}^{1/2}\otimes K_{\Sigma_2}^{1/2}\otimes K_{\Sigma_3}^{1/2}\right)\Big|_p$ to evaluate the wavefunction overlap. This choice, analogous to the choice of Ω in (4.23), introduces the same scaling with ℓ.

Once we introduce four-dimensional chiral superfields $\{\phi_i\}$ with canonical kinetic terms, the F-terms become

$$L_F = \lambda_{ijk}^0 \int d^2\theta \frac{\phi_i\phi_j\phi_k}{\sqrt{M_*^2{\rm Vol}(\Sigma_i)\, M_*^2{\rm Vol}(\Sigma_j)\, M_*^2{\rm Vol}(\Sigma_k)}}\,. \tag{4.26}$$

We note that when all matter curves have comparable volumes set by the overall size of ${\rm Vol}(S)$, ${\rm Vol}(\Sigma)^2 \sim {\rm Vol}(S)$. In this case, (4.8) implies:

$$L_F = \alpha_{\rm GUT}^{3/4}\lambda_{ijk}^0 \int d^2\theta\, \phi_i\phi_j\phi_k. \tag{4.27}$$

In rescaling each field by an appropriate power of the volume factor, we shall typically use the classical value of ${\rm Vol}(\Sigma_i)$. Strictly speaking, this approximation is only valid in the supergravity limit. Due to the fact that in F-theory there is at present no perturbative treatment of quantum corrections, most of the numerical results obtained throughout this paper can only be reliably treated as order of magnitude estimates.

5. General overview of the models

In this section we provide a guide to the class of models we study. The choice of Kähler surface S already determines many properties of the low energy effective theory. In keeping

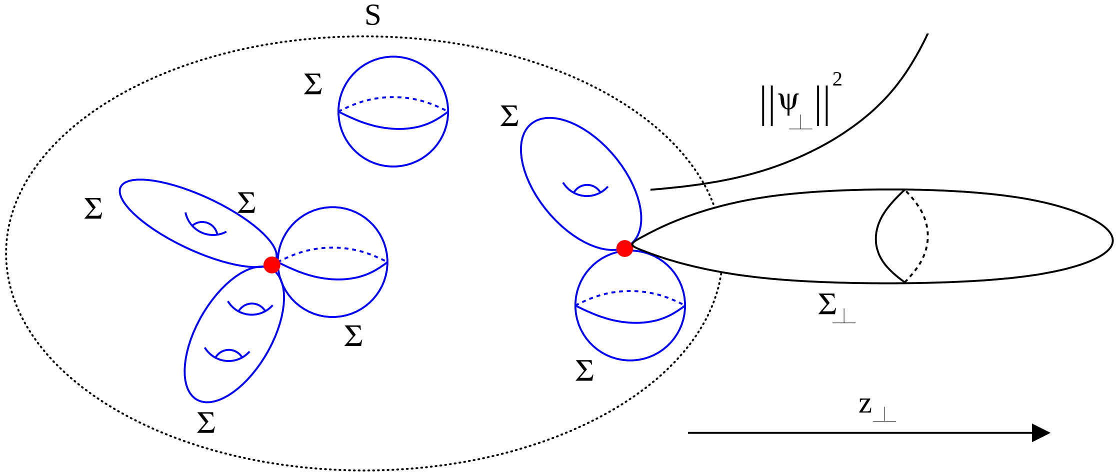

Figure 3: The bulk group on the Kähler surface S corresponds to a singularity of type G_S. Over complex codimension one matter curves in S which we denote by Σ, this singularity type can further enhance so that six-dimensional matter fields localize on these curves. Over complex codimension two points in S the singularity type can enhance further. On the left of the figure we depict a triple intersection of matter curves in S. It is also possible for one of the matter curves to intersect S at a point. Depending on the background gauge fluxes and local curvatures, wave functions localized on curves normal to the GUT brane are either exponentially suppressed or of order one near the point of contact with the GUT brane.

with our general philosophy, we require that the spectrum at low energies must not contain any exotics. When $h^{1,0}(S) \neq 0$, we expect the low energy spectrum to contain additional states obtained by reduction of the bulk supergravity modes of the compactification. For this reason we shall always require that $\pi_1(S)$ is a finite group. There are two further possible refinements depending on whether or not the model in question admits a limit in which M_{GUT} remains finite while $M_{\text{pl}} \to \infty$. In order to fully decouple gravity, the extension of the local metric on S to a local Calabi-Yau fourfold must possess a limit in which the surface S can shrink to zero size. In particular, this imposes the condition that K_S^{-1} must be ample. This is equivalent to the condition that S is a del Pezzo surface, in which case $h^{2,0}(S) = 0$. We note that the degree $n \geq 2$ Hirzebruch surfaces satisfy $h^{2,0}(S) = 0$ but do not define fully consistent decoupled models.

In fact, even the way in which the gauge group of the GUT breaks to that of the MSSM strongly depends on whether or not such a decoupling limit exists. For surfaces with $h^{2,0}(S) \neq 0$, the zero mode content will contain contributions from the bulk holomorphic $(2,0)$ form. Because the $(2,0)$ form determines the position of the exceptional brane inside of the threefold base B_3, a non-zero vev for the associated zero modes corresponds to the usual breaking of the GUT group via an adjoint-valued chiral superfield.[8] Along these lines, we present some examples of four-dimensional GUT models which can originate from surfaces of general type. *An important corollary of this condition is that the usual four-*

[8]The potential application of this GUT breaking mechanism was noted in a footnote of [52] and has also been discussed in [15, 16].

dimensional field theory GUT models cannot be fully decoupled from gravity! We believe this is important because it runs counter to the usual effective field theory philosophy that issues pertaining to the Planck scale can always be decoupled. This is in accord with the existence of a swampland of effective field theories which may not admit a consistent UV completion which includes gravity [53]. Moreover, as we explain in greater detail later, it is also possible that a generic surface of general type may not support sufficiently many matter curves of the type needed to engineer a fully realistic four-dimensional GUT model.

When available, discrete Wilson lines in higher-dimensional theories provide another way to break the GUT group to $G_{\rm std}$. Indeed, most models based on compactifications of the heterotic string on Calabi-Yau threefolds require discrete Wilson lines to break the gauge group and project out exotics from the low energy spectrum. When $\pi_1(S) \neq 0$, a similar mechanism for gauge group breaking is available for exceptional seven-brane theories. As an example, we present a toy model where S is an Enriques surface and $G_S = \mathrm{SU}(5)$. In our specific example, we find that the zero mode content contains additional vector-like pairs of fields in exotic representations of $G_{\rm std}$.

We next turn to the primary case of interest for bottom up string phenomenology where S is a del Pezzo surface. Because $h^{2,0}(S) = 0$ and $\pi_1(S) = 0$ for del Pezzo surfaces, the two mechanisms for gauge group breaking mentioned above are now unavailable. In this case, the GUT group breaks to a smaller subgroup due to non-trivial internal fluxes. For example, the group SU(5) can break to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$ when the internal flux takes values in the $\mathrm{U}(1)_Y$ factor. In heterotic compactifications this mechanism is unavailable because a non-zero internal field strength would generate a string scale mass for the U(1) hypercharge gauge boson in four dimensions [23]. We find that in F-theory compactifications *without a heterotic dual*, there is a natural topological condition for the four-dimensional gauge boson to remain massless. Our expectation is that this condition is satisfied for many choices of compact threefolds B_3. In the remainder of this section we discuss further properties of del Pezzo models.

Along these lines, we present models based on $G_S = \mathrm{SU}(5)$ where the gauge group of the eight-dimensional theory breaks directly to $G_{\rm std}$ in four dimensions, as well as a hybrid scenario where $G_S = \mathrm{SO}(10)$ breaks to $\mathrm{SU}(5) \times \mathrm{U}(1)$ in four dimensions and then subsequently descends from a flipped SU(5) GUT model to the MSSM. In fact, we also present a general no go theorem showing that direct breaking of SO(10) to $G_{\rm std}$ via abelian fluxes always generates extraneous matter in the low energy spectrum. In both the regular SU(5) and flipped SU(5) scenarios, we find that in order to achieve the exact spectrum of the MSSM, all of the matter fields must localize on Riemann surfaces. In the $G_S = \mathrm{SU}(5)$ models, the matter fields organize into the $\overline{5}$ and 10 of SU(5). In the $G_S = \mathrm{SO}(10)$ models, a complete multiplet in the 16 of SO(10) localizes on the matter curves. In both cases, all matter localizes on curves so that all of the tree level superpotential terms descend from the triple intersection of matter curves. When some of the matter localizes on different curves, this leads to texture zeroes in the Yukawa matrices.

In addition to presenting some examples of minimal del Pezzo models, one of the primary purposes of this paper is to develop a number of ingredients which can be of use in further more refined model building efforts. A general overview of these ingredients has already been given in the Introduction, so rather than repeat this here, we simply summarize the primary themes of the minimal SU(5) model which recur throughout this paper. The most prominent ingredient is the internal hypercharge flux which facilitates GUT breaking. This hyperflux also provides a natural solution to the doublet-triplet splitting problem and generates distorted GUT mass relations for the lighter generations. More generally, the presence of additional global U(1) symmetries in the low energy theory forbids a number of potentially problematic interaction terms from appearing in the superpotential. Topologically, the absence of dangerous operators translates into conditions on how the matter curves intersect inside of S. For example, proton decay is automatically suppressed when the Higgs up and down fields localize on different matter curves. When these curves do not intersect, the μ term is zero. When the Higgs matter curves do intersect, the resulting μ term can be naturally suppressed. Indeed, an important feature of all the models we consider is that while expectations from effective field theory would suggest that vector-like pairs will always develop a suitably large mass, here we find two distinct possibilities depending on the choice of the sign for the gauge fluxes: In one case (when the normal wave function is attracted to our brane) we essentially recover the field theory intuition. On the other hand, with a different choice of sign (when the normal wave function is repelled from our brane) we find the opposite situation, where μ is highly suppressed. The ostensibly large mass term corresponding to the vev of a gauge singlet is in fact exponentially suppressed since its wave function is very small near our brane. Here, the principle of decoupling is especially important because the large positive curvature of the del Pezzo surface can lead to a natural suppression of the normal wave functions. This provides an explanation for why the μ term is far below the GUT scale, as well as why the neutrino masses are so far below the electroweak scale. While we discuss many of these mechanisms in the specific context of the minimal SU(5) model, these same features carry over to the flipped SU(5) GUT models as well. In such cases, additional well-established field theoretic mechanisms are also available. For example, four-dimensional flipped SU(5) models already contain an elegant mechanism for doublet-triplet splitting which also naturally suppresses dangerous dimension five operators responsible for proton decay. In this case, we can also utilize a conventional seesaw mechanism to generate hierarchically light neutrino masses.

6. Surfaces of general type

In this section we present some examples of models where Planck scale physics cannot be decoupled from local GUT models. Recall that in a traditional four-dimensional GUT, the GUT group breaks to G_{std} when an adjoint-valued chiral superfield develops a suitable vev. In F-theory, this requires that the seven-brane wraps a surface with $h^{2,0}(S) \neq 0$. Before proceeding to a discussion of GUT models based on such surfaces, we first discuss some important constraints on matter curves and supersymmetric gauge field configurations for such surfaces.

In many cases, some of the chiral fields of the low energy theory will localize on matter curves in S. When $h^{2,0}(S) \neq 0$, the number of available matter curves will typically be much smaller than the dimension of $H_2(S,\mathbb{Z})$ would suggest. To see this, suppose that an element of $H_2(S,\mathbb{Z})$ corresponds to a holomorphic curve Σ in S. We shall also refer to the class $[\Sigma]$ as an "effective" divisor. Given a $(2,0)$ form Ω on S, note that:

$$\int_{\Sigma}\Omega = \int_{S}\Omega \wedge PD(\Sigma) = 0 \tag{6.1}$$

where $PD(\Sigma)$ denotes the element of $H^2(S,\mathbb{Z})$ which is Poincaré dual to Σ. This last equality follows from the fact that $PD(\Sigma)$ corresponds to the first Chern class of an appropriate line bundle and therefore is of type $(1,1)$.[9] We thus see that although the condition $h^{2,0}(S) \neq 0$ is satisfied by a large class of vacua, at generic points in the complex structure moduli space each element of $H^{2,0}(S,\mathbb{C})$ imposes an additional constraint of the form given by equation (6.1). At the level of cohomology, the divisor classes are parameterized by the Picard lattice of S:

$$Pic(S) = H^{1,1}(S,\mathbb{C}) \cap H^2(S,\mathbb{Z}). \tag{6.2}$$

For example, we note that for a generic algebraic $K3$ surface, $Pic(S)$ has rank one. Indeed, this lattice is generated by the hyperplane class inherited from the projective embedding of a general quartic in $\mathbb{P}^3$. It is only at special points in the complex structure moduli space that additional holomorphic curves are present. An example of a $K3$ surface of this type occurs when the quartic is of Fermat type. In this case, the rank of $Pic(S)$ is instead 20. Because there is a one to one correspondence between line bundles and divisors on S, we conclude that a similar condition holds for the available line bundles on a generic surface.

Having stated these caveats on what we expect for generic surfaces of general type, we now construct an SO(10) GUT model with semi-realistic Yukawa matrices. In order to have a sufficient number of matter curves, we consider a seven-brane with worldvolume gauge group SO(12) wrapping a surface S defined by the blowup at k points of a degree $n \geq 5$ hypersurface in $\mathbb{P}^3$ with n odd. Some properties of hypersurfaces in $\mathbb{P}^3$ are reviewed in appendix B. We have introduced these blown up curves in order to simplify several properties of our example. Indeed, as explained around equation (6.2), the Picard lattice of a surface may have low rank. An important point is that some of the numerical invariants such as $h^{2,0}(S)$ and $\chi(S,\mathcal{O}_S)$ of the degree n hypersurface remain invariant under these blowups. Thus, for many purposes we will be able to perform many of our calculations of the zero mode content as if the surface were a degree n hypersurface in $\mathbb{P}^3$.

For $n \geq 5$, we expect to find a large number of additional adjoint-valued chiral superfields. Geometrically, the vevs of these fields correspond to complex structure moduli in the Calabi-Yau fourfold which can develop a mass in the presence of a suitable background flux. We show that in the present context, a suitable profile of vevs can simultaneously break the GUT group and lift all excess fields from the low energy spectrum.

As explained in section 3, in the context of a local model, we are free to specify the enhancement type along codimension one matter curves inside of S. We first introduce four curves $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_B$ where the singularity type enhances to E_7 so that a half-hypermultiplet in the 32 of SO(12) localizes on each curve. With notation as in appendix B,

[9]This last correspondence follows from the link between divisors and line bundles.

the homology class of each curve is:

$$[\Sigma_1] = E_2 \tag{6.3}$$

$$[\Sigma_2] = E_4 \tag{6.4}$$

$$[\Sigma_3] = E_6 \tag{6.5}$$

$$[\Sigma_B] = -a_1 l_1 - E_8 - E_9. \tag{6.6}$$

where we have written $K_{H_n} = a_1 l_1 + a_2 l_2 + \ldots$ for some generators l_i of $H_2(H_n, \mathbb{Z})$ such that $l_i \cdot l_j = 0$ for $i \neq j$. Using the genus formula $C \cdot (C + K_S) = 2g - 2$, we conclude that the genera of $\Sigma_1, \Sigma_2, \Sigma_3$ are all zero while Σ_B has genus one. We note that in order for Σ_B to represent a holomorphic curve, it may be necessary to go to some special points in the moduli space of the surface S. In the presence of a suitable internal flux, a single generation in the 16 of SO(10) will localize on each of the Σ_i's. The fields localized on Σ_B will instead develop a suitable vev to lift extraneous matter from the low energy spectrum.

We next introduce the curve Σ_R where the singularity type enhances to SO(14) so that a hypermultiplet transforming in the 12 of SO(12) localizes on this curve. The homology class of Σ_R is:

$$[\Sigma_R] = -a_2 l_2 - E_{10} - E_{11} \tag{6.7}$$

so that Σ_R has genus one.

A supersymmetric U(1) gauge field configuration can simultaneously break SO(12) to $\mathrm{SO}(10) \times \mathrm{U}(1)_{\mathrm{PQ}}$ and also induce a net chiral matter content in the four-dimensional effective theory. Representations of SO(12) decompose under the subgroup $\mathrm{SO}(10) \times \mathrm{U}(1)_{\mathrm{PQ}}$ as:

$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1)_{\mathrm{PQ}} \tag{6.8}$$

$$66 \to 45_0 + 1_0 + 10_2 + 10_{-2} \tag{6.9}$$

$$32 \to 16_1 + \overline{16}_{-1} \tag{6.10}$$

$$12 \to 1_2 + 1_{-2} + 10_0. \tag{6.11}$$

All candidate Higgs fields in the 10_{-2} are equally charged under the group $\mathrm{U}(1)_{\mathrm{PQ}}$ and we shall therefore loosely refer to it as a Peccei-Quinn symmetry. We consider configurations such that one generation in the 16_1 of SO(10) localizes along each Σ_i for $i = 1,2,3$. In addition to the matter content of the MSSM, we shall also require that there is extra vector-like matter in the 16_1 and $\overline{16}_{-1}$ localized along Σ_B and a 10_0 and 1_2 localized along Σ_R. When the extra vector-like 16's develop a vev at suitably large energy scales, they will remove an additional $\mathrm{U}(1)_{B-L}$ gauge boson from the low energy spectrum. Further, interaction terms between the 10_0 and 1_{-2} can also serve to remove extraneous matter from the spectrum.

The above requirements are satisfied by a large class of supersymmetric line bundles. For concreteness, we consider the line bundle:

$$L = \mathcal{O}_S(E_1 - E_2 + E_3 - E_4 + E_5 - E_6 - E_{10} + E_{12} + N(E_{14} - E_{15})) \tag{6.12}$$

where to simplify some cohomology calculations, we shall sometimes take N to be a large integer. By inspection, there exists a parametric family of Kähler classes such that the condition:

$$\omega \wedge c_1(L) = 0 \tag{6.13}$$

holds. In the above, ω denotes a particular choice of Kähler form on S.

6.1 Bulk matter content

While all of the chiral matter of the MSSM localizes on the matter curves Σ_1, Σ_2 and Σ_3, the internal U(1) flux specified by the line bundle of equation (6.12) will also induce additional bulk zero modes. The bulk matter content all descends from the adjoint representation of SO(12). First consider the number of chiral superfields transforming in the representation $45_0 + 1_0$. These fields are neutral under $U(1)_{PQ}$ so that the total number of chiral superfields transforming in this representation is $h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S)$. In the present case, $h^1(S, \mathcal{O}_S) = 0$ so that it is enough to compute $h^2(S, \mathcal{O}_S) = h^{2,0}(S)$. The Hodge numbers of S are computed in appendix B with the end result:

$$\left(\frac{1}{6}(n^3 - 6n^2 + 11n) - 1\right) \times (45_0 + 1_0) \in H^2_{\overline{\partial}}(S, \mathcal{O}_S). \tag{6.14}$$

When these fields develop a suitable vev, the GUT group will break to G_{std}.

The chiral superfields transforming in the $10_{\pm 2}$ are classified by the bundle-valued cohomology groups:

$$10_{\pm 2} \in H^0_{\overline{\partial}}(S, L^{\mp 2})^* \oplus H^1_{\overline{\partial}}(S, L^{\pm 2}) \oplus H^2_{\overline{\partial}}(S, L^{\mp 2})^*. \tag{6.15}$$

Now, when the integer N of equation (6.12) is sufficiently large, both $H^0_{\overline{\partial}}(S, L^{\mp 2})^*$ and $H^2_{\overline{\partial}}(S, L^{\mp 2})^*$ will indeed vanish. The resulting dimension of $H^1_{\overline{\partial}}(S, L^{\pm 2})$ can then be computed via an index formula:

$$h^1(S, L^{\pm 2}) = -\left(\chi(\mathcal{O}_S) + \frac{1}{2}c_1(S) \cdot c_1(L^{\pm 2}) + \frac{1}{2}c_1(L^{\pm 2})^2\right) \tag{6.16}$$

$$= -\frac{1}{6}(n^3 - 6n^2 + 11n) + (16 + 4N^2) \tag{6.17}$$

so that there are an equal number of 10_{+2} and 10_{-2}'s. Based on their coupling to the fields localized along the matter curve, we shall tentatively identify these as Higgs fields.

6.2 Localized matter content

We now study the chiral matter content localized on matter curves. By construction, L restricts to a degree one line bundle on the genus zero matter curves $\Sigma_1, \Sigma_2, \Sigma_3$ so that a single generation transforming in the 16_1 localizes on each matter curve. Further, L restricts to a trivial line bundle on Σ_B so that a single vector-like pair of 16_1 and $\overline{16}_{-1}$ localizes along Σ_B. Finally, L restricts to a degree -1 bundle, $\mathcal{O}_{\Sigma_R}(-p)$ on the genus one matter curve Σ_R where p denotes a degree one divisor of Σ_R. In order to achieve one copy of the 10_0, we also include a contribution to the flux from the other seven-brane intersecting the GUT model seven-brane along Σ_R so that $L'_{\Sigma_R} = \mathcal{O}_{\Sigma_R}(p')$, where p' is another degree one divisor of Σ_R. The total field content on Σ_R is therefore given by one $10_{0,1}$, three $1_{-2,1}$'s and one $1_{-2,-1}$, where the two subscripts indicate the U(1) charge with respect to the two U(1) factors.[10] The representation content and type of matter curve are summarized in table 1.

[10] As we explain later in section 10, the overall normalization of the U(1) charges is somewhat inconsequential so long as the fields transform in mathematically well-defined line bundles.

SO(10) Model	Curve	Class	g_Σ	L_Σ	L'^n_Σ
$1 \times 16_1$	Σ_1	E_2	0	$\mathcal{O}_{\Sigma_1}(1)$	$\mathcal{O}_{\Sigma_1}$
$1 \times 16_1$	Σ_2	E_4	0	$\mathcal{O}_{\Sigma_2}(1)$	$\mathcal{O}_{\Sigma_2}$
$1 \times 16_1$	Σ_3	E_6	0	$\mathcal{O}_{\Sigma_3}(1)$	$\mathcal{O}_{\Sigma_3}$
$1 \times \left(16_1 + \overline{16}_{-1}\right)$	Σ_B	$-a_1 l_1 - E_8 - E_9$	1	$\mathcal{O}_{\Sigma_B}(0)$	$\mathcal{O}_{\Sigma_B}$
$1 \times 10_{0,1} + 3 \times 1_{-2,1} + 1 \times 1_{-2,-1}$	Σ_R	$-a_2 l_2 - E_{10} - E_{11}$	1	$\mathcal{O}_{\Sigma_R}(-p)$	$\mathcal{O}_{\Sigma_R}(p')$

Table 1: Representation content, type of matter curve and line bundle assignments for an SO(10) model based on a surface of general type.

As will be clear when we discuss the high energy superpotential, although the $1_{-2,-1}$ couples non-trivially with the $10_{0,1}$ to bulk modes on S, the $1_{-2,1}$'s do not contribute to the cubic superpotential, and we shall therefore neglect their contribution to the low energy theory. To simplify notation, we shall therefore refer to the $10_{0,1}$ as the 10_0 and the $1_{-2,-1}$ as the 1_{-2}.

6.3 High energy superpotential

In the present model, the Yukawa couplings of the MSSM originate from purely bulk couplings and couplings between bulk gauge fields and matter fields localized along matter curves. In addition, a background flux configuration in the Calabi-Yau fourfold will also couple to the complex structure moduli of the compactification. Indeed, as shown in [15], the vevs of the bulk $(2,0)$ form and fields localized along matter curves all determine complex deformations of the background compactification. In the case of fields localized along the matter curve, this corresponds to the "mesonic" branch of moduli space. We therefore conclude that fluxes can induce a non-trivial mass and vev for the corresponding fields. At energy scales close to M_{GUT} but below the energy scale where the first Kaluza-Klein mode can contribute an appreciable amount, the high energy superpotential is:

$$W_{\text{high}} = W_S + W_{S\Sigma\Sigma} + W_{\text{flux}} + W_{\text{np}} \tag{6.18}$$

where:

$$W_S = f_{iIJ} 10^{(I)}_{+2} \times 10^{(J)}_{-2} \times (45^{(i)}_0 + 1^{(i)}_0) \tag{6.19}$$

$$W_{S\Sigma\Sigma} = \lambda_{aJ} 16^{(a)}_1 \times 16^{(a)}_1 \times 10^{(J)}_{-2} + \alpha_a 10^{(a)}_2 \times 10_0 \times 1_{-2} \tag{6.20}$$

$$+ \left(\beta_J 16_1 \times 16_1 \times 10^{(J)}_{-2} + \gamma_J \overline{16}_{-1} \times \overline{16}_{-1} \times 10^{(J)}_2\right) \tag{6.21}$$

$$W_{\text{flux}} = \int\limits_{CY_4} \Omega \wedge G_4 \tag{6.22}$$

$$W_{np} = \mu^{(IJ)}_{-4} 10^{(I)}_{+2} \times 10^{(J)}_{+2} + \mu^{(IJ)}_{+4} 10^{(I)}_{-2} \times 10^{(J)}_{-2} \tag{6.23}$$

In the above, terms proportional to the coefficients λ_{aJ} descend from the three matter curves $\Sigma_1, \Sigma_2, \Sigma_3$, while terms proportional to β_J and γ_J descend from the matter curve

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Σ_B. Here, we have also included the effects of non-perturbatively generated mass terms for the 10's which explicitly violate the $U(1)_{PQ}$ global symmetry. Such terms can originate from exponentially suppressed higher-dimensional operators which couple the fields of the GUT model to additional GUT group singlets. When these singlets develop a suitable vev, they can generate terms of the type given by W_{np}. In this case, the resulting μ term will naturally be exponentially suppressed. A similar mechanism has been analyzed in the context of type II intersecting D-brane models as a potential solution to the μ problem [54].

While stabilizing the moduli in a realistic compactification is certainly a non-trivial task, in a local model, the vevs of the complex structure moduli can effectively be tuned to an arbitrary value. Letting $\Omega^{(0)}$ denote the value of the holomorphic four form of the Calabi-Yau fourfold with the desired values of the complex structure moduli, we note that the critical points of W_{flux} with $G_4 = \lambda(\Omega^{(0)} + \overline{\Omega}^{(0)})$ will indeed yield such a configuration. For compact models, this must be appropriately adjusted because the potential for the overall volume of the Calabi-Yau fourfold will develop a non-supersymmetric minimum.

6.4 Low energy spectrum

We now show that an appropriate choice of vevs in W_{high} given by equation (6.18) can yield a low energy effective theory with precisely the matter content of the MSSM and a semi-realistic low energy superpotential. We first demonstrate that the above model can indeed remove all excess matter at sufficiently high energies. To this end, first note that when a $45_0^{(i)}$ develops the vev:

$$\langle 45_0 \rangle = i\sigma_y \otimes diag(a, a, a, b, b) \tag{6.24}$$

the resulting gauge group will break to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$. By inspection of equation (6.19), when $a \sim M_{\text{GUT}}$, this vev will also remove the Higgs triplets of $10_{-2}^{(J)}$ (and the $10_{+2}^{(I)}$'s) from the low energy spectrum. When the zero mode content contains at least two 45's which have distinct couplings to the product $10_{+2}^{(I)} \times 10_{-2}^{(J)}$, a suitable choice of b for each 45 can be arranged so that at most one pair of $SU(2)_L$ doublets from one linear combination of the 10_{-2}'s will remain massless. We note that this is simply a variant on the well-known Dimopoulos-Wilczek mechanism for achieving doublet-triplet splitting in four-dimensional SO(10) GUT models [55, 56].

In the absence of other field vevs, the resulting spectrum would contain two $SU(2)_L$ doublets from a bulk 10_{-2} as well as its counterpart 10_2. In fact, we now demonstrate that when the flux induces a suitably large mass term for the 10_0 as well as a vev for the 1_{-2}, the resulting low energy spectrum will not contain any fields transforming in the representation 10_2. With the above choice of fluxes, the mass matrix for the 10_0 and remaining 10_2 is schematically of the form:

$$W_{\text{eff}} \supset \begin{bmatrix} 10_2 & 10_0 \end{bmatrix} \begin{bmatrix} 0 & \langle 1_{-2} \rangle \\ \langle 1_{-2} \rangle & \langle M_{\text{flux}} \rangle \end{bmatrix} \begin{bmatrix} 10_2 \\ 10_0 \end{bmatrix} \tag{6.25}$$

so that all extraneous 10_{+2}'s can indeed lift from the low energy spectrum.

The resulting spectrum is almost that of the MSSM at low energies. The only additional matter content is an additional $\mathrm{U}(1)_{B-L}$ gauge boson and a vector-like pair of matter fields 16_1 and $\overline{16}_{-1}$ localized on Σ_B. In fact, when the 16_1 and $\overline{16}_{-1}$ develop a suitable vev, they will break $\mathrm{U}(1)_{B-L}$.

Maximally utilizing conventional four-dimensional field theoretic mechanisms to achieve the correct matter spectrum, this model yields the spectrum of the MSSM at low energies. Moreover, by placing the three generations on three distinct matter curves, a large hierarchy in scales can be generated by a suitable choice of Kähler class.

The effective superpotential is now schematically of the form:

$$W_{\text{eff}} = \mu H_u H_d + \lambda^u_{ij} Q^i U^j H_u + \lambda^d_{ij} Q^i D^j H_d + \lambda^l_{ij} L^i E^j H_d + \lambda^\nu_{ij} L^i N^j_R H_u + \dots \tag{6.26}$$

where the λ_{ij}'s are all diagonal.

While it is of course possible to further refine the above model, we believe this provides a fruitful starting point for analyzing how traditional four-dimensional GUT models can embed in F-theory. Again, we emphasize that strictly speaking, a purely four-dimensional effective field theory approach breaks down in this case because no decoupling limit between M_{GUT} and M_{pl} is available.

7. Surfaces with discrete Wilson lines

In the previous section we presented an example of a four-dimensional GUT model which breaks to the MSSM when a collection of adjoint-valued chiral superfields develop appropriate vevs. This requires that the surface S wrapped by the seven-brane satisfies $h^{2,0}(S) \neq 0$. When $\pi_1(S) \neq 0$, it is also possible for the GUT group to spontaneously break to the gauge group of the Standard Model via an appropriate choice of Wilson lines. In this section we describe some features of models based on the case where S is an Enriques surface. After reviewing some basic properties of such surfaces, we present a toy model with bulk gauge group $G_S = \mathrm{SU}(5)$. Although the correct matter content of the MSSM can localize on matter curves, we find that the discrete Wilson lines also generically produce additional vector-like pairs of zero modes transforming in exotic representations of G_{std}. This can be traced back to the fact that the universal cover of an Enriques surface is a $K3$ surface. Although we do not present a complete model based on an Enriques surface, we discuss how these problems can be avoided by including further field-theoretic mechanisms to lift extraneous matter from the low energy spectrum. It is also possible that other surfaces with different fundamental groups may provide additional possibilities. To this end, we conclude by mentioning some other surfaces which have been studied in the mathematics literature.

We begin by reviewing some relevant features of Enriques surfaces. Further details can be found in [57]. An Enriques surface S is defined by the conditions:

$$K_S^2 = \mathcal{O}_S \text{ but } K_S \neq \mathcal{O}_S \tag{7.1}$$

and that the "irregularity" $h^{1,0}(S) = q(S) = 0$. The non-vanishing Hodge numbers of an Enriques surface are $h^{1,1}(S) = 10$ and $h^{0,0}(S) = h^{2,2}(S) = 1$. The fundamental group of

S is $\pi_1(S) = \mathbb{Z}_2$. Moreover, the universal cover of S is a $K3$ surface. Indeed, the Hodge number $h^{2,0}(K3) = 1$ does not survive in the quotient space. Nevertheless, we shall see that in the presence of discrete Wilson lines, the zero mode content retains some imprint from the underlying $K3$ surface.

Recall that for a $K3$ surface, the intersection form on $H^2(K3, \mathbb{Z})$ is isomorphic to:

$$H^2(K3, \mathbb{Z}) = (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U \tag{7.2}$$

where $-E_8$ denotes minus the intersection form for the Lie algebra E_8 and the "hyperbolic element" U is the intersection form with entries given by the Pauli matrix σ_x. The intersection form on S is instead given by:

$$H^2(S, \mathbb{Z})/Tor = (-E_8) \oplus U \tag{7.3}$$

where in the above we have modded out by possible torsional elements. As an integral lattice, $H^2(S, \mathbb{Z})$ is isomorphic to:

$$H^2(S, \mathbb{Z}) \simeq \mathbb{Z}^{10} \oplus \mathbb{Z}_2. \tag{7.4}$$

We label the generators of $H^2(S, \mathbb{Z})$ as $\alpha_1, \ldots, \alpha_8$ in correspondence with the roots of E_8 and d_1 and d_2 for the generators associated with U such that $d_i \cdot d_j = 1 - \delta_{ij}$. Finally, we label the torsion element as t. An important feature of Enriques surfaces is that the Poincaré dual homology classes for d_1 and d_2 both represent holomorphic elliptic curves in S.

We now present a toy model with S an Enriques surface with bulk gauge group $G_S =$ SU(5) which spontaneously breaks to G_{std} due to a discrete Wilson line taking values in the $\text{U}(1)_Y$ factor. The example we shall now present cannot be considered even semi-realistic because in addition to containing exotic matter, the tree level superpotential contains too many texture zeroes. Nevertheless, it illustrates some of the elements which are necessary in more realistic constrictions. To simplify our discussion, we shall emphasize elements unique to having non-trivial discrete Wilson models.

Because bulk modes descend from the adjoint representation of SU(5) and all of the matter of the Standard Model descends from other representations of SU(5), the chiral superfields of the MSSM must localize on matter curves. The generic $G_S = \text{SU}(5)$ singularity enhances to SU(6) along the Higgs and $\overline{5}_M$ matter curves and enhances to SO(10) along the 10_M matter curve. The matter curves and choice of line bundle assignment are given in table 2.

In the above, we have also indicated how each curve lifts to $K3$. In this case both matter curves $\Sigma_M^{(i)}$ lift to the disjoint union of two curves in $K3$ while the Higgs curve Σ_H lifts to a curve which is fixed by the $\mathbb{Z}_2$ involution in $K3$. As an explicit example, we can consider the case where the covering space of S is a real $K3$ surface and the $\mathbb{Z}_2$ involution corresponds to complex conjugation. In this case, the curve $\Sigma_M^{(i)}$ lifts to a generic holomorphic curve and its image under complex conjugation while Σ_H lifts to a real algebraic curve in $K3$. We now show that in this case the discrete Wilson line projects out the Higgs triplet from the low energy spectrum.

Enriques Model	Curve	$K3$ curve	Class	g_Σ	L_Σ	L'^n_Σ
$1 \times (5_H + \overline{5}_H)$	Σ_H	$\widetilde{\Sigma}_H$	d_1	1	$\mathbb{Z}_2 \otimes \mathcal{O}_{\Sigma_H}$	$\mathcal{O}_{\Sigma_H}$
$3 \times \overline{5}_M$	$\Sigma_M^{(1)}$	$\widetilde{\Sigma}_M^{(1)} \amalg \widetilde{\Sigma}'^{(1)}_M$	d_2	1	$\mathcal{O}_{\Sigma_M^{(1)}}$	$\mathcal{O}_{\Sigma_M^{(1)}}(-3p_1)$
$3 \times 10_M$	$\Sigma_M^{(2)}$	$\widetilde{\Sigma}_M^{(2)} \amalg \widetilde{\Sigma}'^{(2)}_M$	d_1	1	$\mathcal{O}_{\Sigma_M^{(2)}}$	$\mathcal{O}_{\Sigma_M^{(2)}}(3p_1)$

Table 2: Representation content, type of matter curve and line bundle assignments for a model based on an Enriques surface.

Because the Higgs curve is fixed by the $\mathbb{Z}_2$ involution, the fields localized on this curve will transform non-trivially in the presence of a $\mathbb{Z}_2$ Wilson line. The analysis below equation (3.12) applies equally well to fields localized on matter curves. Under the breaking pattern $SU(5) \supset SU(3) \times SU(2) \times U(1)$, the 5 of SU(5) decomposes to $(1,2)_3 + (3,1)_{-2}$. In this case, the relevant cohomology group lifts to the $\mathbb{Z}_2$ odd eigenspace:

$$5_H \in H^0_{\overline{\partial}}(\widetilde{\Sigma}_H, \mathcal{O}_{\widetilde{\Sigma}_H}) \simeq \mathbb{C}_{(-)}. \tag{7.5}$$

Hence, we conclude that the total wave function for the components of the 5_H and $\overline{5}_H$ take values in the invariant subspaces:

$$(1,\overline{2})_{-3}, (1,2)_3 \in \left[\mathbb{C}_{(-)} \otimes H^0_{\overline{\partial}}(\widetilde{\Sigma}_H, \mathcal{O}_{\widetilde{\Sigma}_H})\right]^{\mathbb{Z}_2} \simeq \mathbb{C} \tag{7.6}$$

$$(\overline{3},1)_{-2}, (3,1)_2 \in \left[\mathbb{C}_{(+)} \otimes H^0_{\overline{\partial}}(\widetilde{\Sigma}_H, \mathcal{O}_{\widetilde{\Sigma}_H})\right]^{\mathbb{Z}_2} = 0. \tag{7.7}$$

Hence, the Higgs triplet is absent from the low energy spectrum while the Higgs up and down doublets remain.

The matter content of this example is not fully realistic because it also contains contributions from the bulk zero modes which appear as vector-like pairs transforming in exotic representations of G_{std}. To compute the bulk zero mode content in the presence of the discrete Wilson line, we again apply the analysis below equation (3.12) in the special case where the bundle $\mathcal{T}$ is trivial. Decomposing the adjoint representation of SU(5) to $SU(3) \times SU(2) \times U(1)$, the only irreducible representations which transform non-trivially under the U(1) factor are the $(3,\overline{2})_{-5}$ and $(\overline{3},2)_5$. We now compute the number of bulk zero modes transforming in the $(3,\overline{2})_{-5}$. In the covering $K3$ space, the contribution to the number of zero modes from the holomorphic $(2,0)$ is given by the $\mathbb{Z}_2$ invariant subspace:

$$(3,\overline{2})_{-5} \in \left[\mathbb{C}_{(-)} \otimes H^2_{\overline{\partial}}(K3, \mathcal{O}_{K3})\right]^{\mathbb{Z}_2} \tag{7.8}$$

where the $\mathbb{C}_{(-)}$ factors indicates the charge of the representation $(3,\overline{2})_{-5}$ under the $\mathbb{Z}_2$ subgroup of $U(1)_Y$. Next recall that the $\mathbb{Z}_2$ group action on the holomorphic $(2,0)$ form sends $\varphi \mapsto -\varphi$. In particular, this implies that the cohomology group $H^2_{\overline{\partial}}(K3, \mathcal{O}_{K3}) \simeq \mathbb{C}_{(-)}$. A similar analysis also holds for zero modes transforming in the representation $(3,\overline{2})_{-5}$. Because $\mathbb{C}_{(-)} \otimes \mathbb{C}_{(-)}$ is $\mathbb{Z}_2$ invariant, we conclude that the low energy spectrum contains exotic vector-like pairs.

There are potentially several ways to avoid the presence of these exotics. For example, when $G_S = \mathrm{SO}(10)$, a combination of U(1) flux breaking and discrete Wilson line breaking might avoid any contributions from bulk zero modes. Moreover, even if additional exotic particles are present in the low energy spectrum, it is conceivable that an appropriately engineered superpotential could cause these exotics to develop a large mass.

It is also possible to consider a more general class of surfaces with non-trivial discrete Wilson lines. In the present context the maximal case of interest would be surfaces with $h^{1,0}(S) = h^{2,0}(S) = 0$ and $\pi_1(S)$ a finite group. Some examples of surfaces such as the classical Godeaux and Campadelli surfaces may be found in [57]. As a technical aside, we note that one particularly interesting class of surfaces can be obtained by choosing n distinct points of a del Pezzo 9 surface and performing an order a_i logarithmic transformation at the i^{th} point.[11] The resulting surface has the same Hodge numbers, Euler character and signature as the del Pezzo 9 surface and is called a Dolgachev surface, $D(a_1, \ldots, a_n)$. For example, when $n = 2$ and a_1 and a_2 have a common divisor, the fundamental group is $\pi_1(D(a_1, a_2)) \simeq \mathbb{Z}_m$ where $m = \gcd(a_1, a_2)$. See [58, 59] and references therein for further discussion of Dolgachev surfaces defined by two logarithmic transformations. We note that the case $a_1 = 2$, $a_2 = 2$ corresponds to the Enriques surface. It is also common in the mathematics literature to treat the more general case as well. When the a_i are pairwise co-prime integers, the resulting fundamental group is [60]:

$$\pi_1(D(a_1, \ldots, a_n)) = \langle t_1, \ldots, t_n | t_i^{a_i} = 1, t_1 \cdots t_n = 1 \rangle \,. \tag{7.9}$$

Given the prominent role that the del Pezzo 9 surface has played in recent heterotic models such as [8, 9], it would be interesting to study models based on such Dolgachev surfaces.

8. Geometry of del Pezzo surfaces

In the remainder of this paper we focus on the case of primary interest where S is a del Pezzo surface. In this case, it is at least in principle possible to consistently decouple the Planck scale from the GUT scale. Because much of the analysis to follow relies on properties of del Pezzo surfaces, in this section we collect various relevant facts about the geometry of such surfaces. After giving the definition of del Pezzo surfaces, we catalogue the moduli of such surfaces which must be stabilized in a globally consistent model. Next, we review the beautiful connection between the homology groups of del Pezzo surfaces and the root lattices of exceptional Lie algebras. In particular, we show that the line bundles L on S such that both L and L^{-1} have trivial cohomology are in one to one correspondence with the roots of the corresponding exceptional Lie algebra. This classification will prove important when we study vacua with trivial bulk zero mode content.

The two simplest examples of del Pezzo surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. There are eight additional surfaces defined as the blowup of $\mathbb{P}^2$ at up to eight points in general position. We shall refer to such surfaces as del Pezzo N (dP_N) surfaces for the case of N blown up points.

We now describe the Kähler and complex structure moduli spaces of these surfaces. First consider the Kähler moduli of del Pezzo surfaces. $\mathbb{P}^1 \times \mathbb{P}^1$ has two Kähler moduli

[11] See [57] for the definition and further properties of logarithmic transformations of surfaces.

corresponding to the volume of the two $\mathbb{P}^1$ factors. There is a single Kähler modulus which fixes the overall size of $\mathbb{P}^2$. In addition to the overall size of the $\mathbb{P}^2$, for the del Pezzo N surfaces, there are N further moduli corresponding to the volume of each blown up cycle. Further properties of the Kähler cone for each del Pezzo surface are reviewed in appendix A of [15].

In addition to the Kähler moduli of each del Pezzo surface, these surfaces may also possess a moduli space of complex structures. For $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ there is a unique choice of complex structure. When $S = dP_N$, the overall $PGL(3)$ symmetry of $\mathbb{P}^2$ implies that the number of complex structure moduli is $2N - 8$ so that in an isolated local model only surfaces with $5 \leq N \leq 8$ possess a moduli space of complex structures. In the context of a globally consistent moduli, this distinction is somewhat artificial because the overall $PGL(3)$ action on $\mathbb{P}^2$ may not properly extend to the compact threefold base.

We next describe the homology groups of the del Pezzo surfaces. The homology group $H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ is two dimensional and has generators σ_1 and σ_2 corresponding to the two $\mathbb{P}^1$ factors. These generators have intersection product:

$$\sigma_i \cdot \sigma_j = 1 - \delta_{ij} \tag{8.1}$$

where δ_{ij} is the Kronecker delta. The canonical class for $\mathbb{P}^1 \times \mathbb{P}^1$ is:

$$K_{\mathbb{P}^1 \times \mathbb{P}^1} = -c_1(\mathbb{P}^1 \times \mathbb{P}^1) = -2\sigma_1 - 2\sigma_2. \tag{8.2}$$

In particular, $-K_{\mathbb{P}^1 \times \mathbb{P}^1}$ defines a Kähler class on $\mathbb{P}^1 \times \mathbb{P}^1$ where both $\mathbb{P}^1$ factors have volume two in an appropriate normalization.

The homology group $H_2(dP_N, \mathbb{Z})$ is $N+1$ dimensional and has generators $H, E_1, \ldots, E_N$ where H denotes the hyperplane class inherited from $\mathbb{P}^2$ and the E_i denote the exceptional divisors associated with the blowup. These generators have intersection product:

$$H \cdot H = 1,\ H \cdot E_i = 0,\ E_i \cdot E_j = -\delta_{ij} \tag{8.3}$$

so that the signature of $H_2(dP_N, \mathbb{Z})$ is $(+, -^N)$. The canonical class for dP_N is:

$$K_{dP_N} = -c_1(dP_N) = -3H + E_1 + \ldots + E_N. \tag{8.4}$$

There is a beautiful connection between del Pezzo $N \geq 2$ surfaces and exceptional Lie algebras. This material is reviewed for example in [61] and has played a role in proposed M-theory dualities [62]. We now review how the sublattice of $H_2(dP_N, \mathbb{Z})$ orthogonal to K_{dP_N} is identified with the root space of the corresponding Lie algebra E_N. Because dP_2 admits a different treatment, first consider the dP_N surfaces with $N \geq 3$. The generators of the lattice $\langle K_{dP_N} \rangle^{\perp}$ are:

$$\alpha_1 = E_1 - E_2, \ldots, \alpha_{N-1} = E_{N-1} - E_N, \alpha_N = H - E_1 - E_2 - E_3. \tag{8.5}$$

The intersection product of the α_i's is identical to minus the Cartan matrix for the dot product of the simple roots for the corresponding Lie algebra E_N. For dP_2, the single generator of the lattice $\langle K_{dP_N} \rangle^{\perp}$ is given by $E_1 - E_2$, which we identify as a root of $su(2)$.

This correspondence further extends to include the Weyl group of the exceptional Lie algebras. In the following we shall adopt a "geometric" convention so that the signature of the root space is negative definite.[12] The Weyl group for a simply connected Lie algebra with simple roots $\alpha_1, \ldots, \alpha_N$ is generated by the Weyl reflections w_{α_i}. Given an element α of the root lattice, the Weyl reflected vector $w_{\alpha_i}(\alpha)$ is:

$$w_{\alpha_i}(\alpha) = \alpha + (\alpha \cdot \alpha_i)\alpha_i. \tag{8.6}$$

This is precisely the action of the large group of diffeomorphisms for the del Pezzo N surfaces on the corresponding generators orthogonal to K_{dP_N}. Indeed, note that the canonical class is invariant under the action of the Weyl group.

Anticipating future applications, we now show that when S is a del Pezzo $N \geq 2$ surface, the collection of all line bundles L such that:

$$H^i_{\overline{\partial}}(S, L^{\pm 1}) = 0 \tag{8.7}$$

for all i are in one to one correspondence with the roots of the Lie algebra E_N.

Because the indices defined by L and L^{-1} must separately vanish, the difference in the two indices also vanishes:

$$0 = \chi(dP_N, L) - \chi(dP_N, L^{-1}) = c_1(dP_N) \cdot c_1(L) = -K_{dP_N} \cdot c_1(L). \tag{8.8}$$

Treating $c_1(L)$ as an element of $H_2(dP_N, \mathbb{Z})$, $c_1(L)$ is therefore a vector in the orthogonal complement of the canonical class. Hence, $c_1(L)$ corresponds to an element of the root lattice of E_N. Utilizing equation (8.8), the index $\chi(dP_N, L)$ now takes the form:

$$\chi(dP_N, L) = 1 + \frac{1}{2} c_1(L) \cdot (c_1(L) + c_1(dP_N)) = 1 + \frac{1}{2} c_1(L) \cdot c_1(L) \tag{8.9}$$

which vanishes provided:

$$c_1(L) \cdot c_1(L) = -2 \tag{8.10}$$

which is the condition for $c_1(L)$ to correspond to a root of E_N. Conversely, we note that given a root α of $\langle K_{dP_N} \rangle^{\perp}$, the line bundle $L = \mathcal{O}_{dP_N}(\alpha)$ defines a supersymmetric gauge field configuration. The vanishing theorem of [63, 15] and the vanishing of the corresponding index now imply that all cohomology groups are trivial.

A similar analysis holds for the remaining del Pezzo surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_1. When $S = \mathbb{P}^2$, we note that because $H_2(\mathbb{P}^2, \mathbb{Z})$ has a single generator given by the hyperplane class of $\mathbb{P}^2$, all non-trivial line bundles L have $c_1(L) \cdot c_1(\mathbb{P}^2) \neq 0$ so that equation (8.8) is never satisfied.

To treat the cases $S = \mathbb{P}^1 \times \mathbb{P}^1, dP_1$ and in order to partially widen the scope of our discussion, we note that these del Pezzo surfaces are also Hirzebruch surfaces. More generally, recall that the middle homology of the degree n Hirzebruch surface $\mathbb{F}_n$ has generators σ and f which have intersection pairing:

$$f \cdot f = 0,\ f \cdot \sigma = 1,\ \sigma \cdot \sigma = -n. \tag{8.11}$$

[12] With this sign convention, a root α satisfies $\alpha \cdot \alpha = -2$.

The canonical class for $\mathbb{F}_n$ is:

$$K_{\mathbb{F}_n} = -c_1(\mathbb{F}_n) = -(n+2)f - 2\sigma. \tag{8.12}$$

We now show that $\mathbb{F}_0$ is the only Hirzebruch surface which admits line bundles satisfying equation (8.7). To this end, consider a line bundle $L = \mathcal{O}_{\mathbb{F}_n}(af + b\sigma)$. In order to satisfy equation (8.7), we must have:

$$0 = \chi(\mathbb{F}_n, L) - \chi(\mathbb{F}_n, L^{-1}) = c_1(\mathbb{F}_n) \cdot c_1(L) = b(n+2) + 2a - 2bn^2. \tag{8.13}$$

When this condition is satisfied, the index $\chi(\mathbb{F}_n, L)$ vanishes provided:

$$0 = \chi(\mathbb{F}_n, L) = 1 + \frac{1}{2}c_1(\mathbb{F}_n) \cdot c_1(L) + \frac{1}{2}c_1(L) \cdot c_1(L) \tag{8.14}$$

$$= 1 + \frac{1}{2}(2ab - b^2n^2) = 1 + \frac{1}{2}(b^2n^2 - b^2(n+2)) \tag{8.15}$$

or,

$$-2 = b^2(n^2 - (n+2)). \tag{8.16}$$

In order for this equation to possess a solution over the integers, $b = \pm 1$ and $n^2 - n = 0$ so that $n = 0$ or $n = 1$. First consider the case where $n = 1$. Returning to equation (8.13), when $n = 1$ and $b = \pm 1$, we find that $a = \pm 1/2$, which is not an integer. We therefore conclude that the only remaining case is $n = 0$. For $\mathbb{F}_0$, the only line bundles satisfying equation (8.7) are $L = \mathcal{O}_{\mathbb{F}_0}(\pm f \mp \sigma) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\pm\sigma_1 \mp \sigma_2)$ where in the final equality we have reverted to the notation of equation (8.1).

9. GUT breaking via U(1) fluxes

When S is a del Pezzo surface, the zero mode content does not contain any adjoint-valued chiral superfields which could potentially play the role of a four-dimensional GUT Higgs fields. In this section we present an alternative mechanism where the GUT group breaks due to non-trivial hypercharge flux in the internal directions. Experience with other string compactifications suggests that a non-trivial internal field strength would cause the photon to develop a string scale mass because this gauge boson couples non-trivially to the p-form gauge potentials of the closed string sector. In this section we present a topological criterion for this U(1) gauge boson to remain massless. This then provides a novel mechanism for GUT group breaking in F-theory.

To analyze whether the coupling to closed string modes will generate a mass for the U(1) gauge boson, first recall that the ten-dimensional supergravity action contains the terms (neglecting the overall normalization of individual terms by order one constants):[13]

$$S^{(10d)} \supset M_*^8 \int_{\mathbb{R}^{3,1} \times B_3} C \,\triangle_{10}\, C - M_*^4 \int_{\mathbb{R}^{3,1} \times S} Tr(F \wedge *_8 F) + M_*^4 \int_{\mathbb{R}^{3,1} \times S} C_{(4)} \wedge Tr(F \wedge F) \tag{9.1}$$

[13] For D-branes, the relative normalizations between these terms contains factors of g_s. In the present class of models, this distinction is ambiguous because these vacua exist in a regime of strong coupling.

where $C_{(4)}$ denotes the RR four-form gauge potential and F denotes the eight-dimensional field strength of the seven-brane. Letting $\langle F_S \rangle$ denote the non-vanishing field strength in the internal directions, integrating out $C_{(4)}$ yields a term in the effective action of the form:

$$S_{\text{eff}}^{(10d)} \supset \int\limits_{\mathbb{R}^{3,1}\times B_3} \delta_{\mathbb{R}^{3,1}\times S} \wedge \langle F_S \rangle \wedge F \frac{1}{\triangle_{10}} \delta_{\mathbb{R}^{3,1}\times S} \wedge \langle F_S \rangle \wedge F \tag{9.2}$$

where $\delta_{\mathbb{R}^{3,1}\times S}$ denotes the delta function for the seven-brane and we have dropped the overall trace because our primary interest is in abelian instanton configurations.

Next, expand $\delta_{\mathbb{R}^{3,1}\times S} \wedge \langle F_S \rangle$ in a basis of eigenmodes so that:

$$\delta_{\mathbb{R}^{3,1}\times S} \wedge \langle F_S \rangle = \sum_{\alpha} f_\alpha \psi_\alpha \tag{9.3}$$

where $\triangle_6 \psi_\alpha = \lambda_\alpha \psi_\alpha$ denote eigenmodes of the Laplacian on B_3 and f_α denote the associated Fourier coefficients. We thus arrive at a non-local term in the four-dimensional effective action:

$$L_{\text{eff}}^{(4d)} \supset \sum_{\alpha} \int_S F \wedge \frac{\left\| f_\alpha \psi_\alpha |_S \right\|^2}{\triangle_4 + \lambda_\alpha} F \tag{9.4}$$

$$= \sum_{\alpha=0} \int_S \left\| f_\alpha \psi_\alpha |_S \right\|^2 A^2 + \sum_{\alpha\neq 0} \int_S F \wedge \frac{\left\| f_\alpha \psi_\alpha |_S \right\|^2}{\triangle_4 + \lambda_\alpha} F. \tag{9.5}$$

so that the contribution from zero modes of Δ_6 induces a mass term for the four-dimensional gauge boson. The remaining modes induce a non-local operator which tends to zero in the decompactification limit.

The zero modes of Δ_6 which can potentially couple to the internal field on S correspond to harmonic representatives of the cohomology group $H^2(B_3, \mathbb{R})$ which are Poincaré dual to elements of $H_4(B_3, \mathbb{Z})$. For concreteness, we let Γ denote such a four-cycle. In the same spirit as [64], we therefore conclude that the four-dimensional U(1) gauge boson will remain massless provided the class in $H^2(S, \mathbb{Z})$ corresponding to $\langle F_S \rangle$ integrates trivially when wedged with any element of $H^2(B_3, \mathbb{Z})$. In other words, given any four-cycle Γ in B_3, Γ must intersect trivially with the Poincaré dual of $\langle F_S \rangle$ which we denote as $[F_S]$ for some element of $H_2(S, \mathbb{Z})$. This implies that the cycle $[F_S]$ must be trivial in B_3.[14] We note that just as in [64], this entire discussion can be phrased in terms of the relative cohomology between S and B_3, and we refer the reader there for more details on this type of argument.

Our expectation is that this condition can be met in a large number of cases. Indeed, in backgrounds where the $(2,0)$ form vanishes, a line bundle L corresponds to a supersymmetric gauge field configuration when [15]:

$$\omega \wedge c_1(L) = 0 \tag{9.6}$$

where ω denotes the Kähler form on S. In particular, if this ω descends from the Kähler form in the threefold base B_3, this is a necessary condition for the Poincaré dual of $\langle F_S \rangle$

[14] This same observation has been made independently by M. Wijnholt.

to lift to a trivial class in $H_2(B_3, \mathbb{Z})$. Note that when $\dim H_2(B_3, \mathbb{Z}) = 1$, this condition is in fact sufficient.

For illustrative purposes, we now show that there exist compactifications of F-theory where this condition can be met. To this end, we consider an elliptically fibered Calabi-Yau fourfold with base $B_3 = \mathbb{P}^3$. In this case, the homology ring $H_*(\mathbb{P}^3, \mathbb{Z})$ is generated by the hyperplane class $H_{\mathbb{P}^3}$. Introducing homogeneous coordinates x_0, x_1, x_2, x_3, we recall that the vanishing locus of a generic degree two polynomial in the x_i defines a $\mathbb{P}^1 \times \mathbb{P}^1$ in B_3, and the vanishing locus of a generic degree three polynomial defines a del Pezzo 6 surface in B_3. As reviewed in appendix B, a multiple of the generator $H_{\mathbb{P}^3}$ restricts to the anti-canonical class of a degree n hypersurface in $\mathbb{P}^3$.

Letting σ_1 and σ_2 denote the generators of $H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ corresponding to the two $\mathbb{P}^1$ factors, the class $\sigma_1 - \sigma_2$ lifts to a trivial class in $\mathbb{P}^3$ due to the fact that $K_{\mathbb{P}^1 \times \mathbb{P}^1} \cdot (\sigma_1 - \sigma_2) = 0$. Similar considerations apply for the del Pezzo 6 surface because all of the two-cycles corresponding to elements in the root lattice of E_6 are orthogonal to K_{dP_6}.

9.1 Absence of a heterotic analogue

Given the usual heterotic/F-theory duality, it is natural to ask whether GUT group breaking via internal fluxes can also occur in the heterotic string. A general obstruction to using U(1) fluxes in heterotic models was already noted in [23]. In fact, in all F-theory models which admit a heterotic dual, the mechanism described above is unavailable! To establish this, first recall that the basic heterotic/F-theory duality relates compactifications of the heterotic string on an elliptic curve to compactifications of F-theory on an elliptically fibered $K3$ [65]. Extending this duality fiberwise, the heterotic string compactified on an elliptically fibered Calabi-Yau threefold is dual to F-theory compactified on a $K3$-fibered Calabi-Yau fourfold. In this case, the threefold base of the F-theory compactification is a $\mathbb{P}^1$ fibration over a Kähler surface S.

We now establish that in this case, an internal hypercharge flux will always cause the corresponding four-dimensional gauge boson to lift from the low energy theory. As explained previously, it is enough to determine whether this internal flux wedges non-trivially with any two forms in $H^2(B_3, \mathbb{R})$. To see why this occurs, first consider the case where the fibration is trivial so that the threefold base is of the form $S \times \mathbb{P}^1 = B_3$. In this case, we note that:

$$H^2(B_3, \mathbb{R}) = H^2(S \times \mathbb{P}^1, \mathbb{R}) \simeq H^2(S, \mathbb{R}) \oplus H^2(\mathbb{P}^1, \mathbb{R}). \tag{9.7}$$

This implies that all non-zero elements of $H^2(S, \mathbb{R})$ wedge non-trivially with some element of $H^2(B_3, \mathbb{R})$. Next consider the case of a non-trivial fibration. The only consequence of the non-trivial fibration structure is that the cohomology group $H^2(B_3, \mathbb{R})$ could potentially contain additional contributions on top of those already present in the product formula of equation (9.7).[15] In particular, all of the elements of the cohomology group of $H^2(S, \mathbb{R})$ again wedge non-trivially with some element of $H^2(B_3, \mathbb{R})$.

[15]At a more formal level, this is a direct consequence of the Leray-Serre spectral sequence.

10. Avoiding exotica

As argued in the previous section, abelian fluxes provide a potentially generic mechanism for breaking the GUT group to G_{std}. As shown in [15], such fluxes also determine the zero mode content of the low energy effective theory. It thus follows that the zero mode content of the theory may not match to the MSSM. In keeping with our general philosophy, we require that all of the zero modes other than the Higgs fields must organize into complete GUT multiplets. Indeed, if these zero modes do not fill out complete GUT multiplets, they can potentially spoil the unification of the gauge couplings.

It is in principle possible that these restrictions can be relaxed. If all exotics come in vector-like pairs, effective field theory arguments would appear to suggest that such pairs will develop a large mass and lift from the low energy spectrum. We note that in the present case, all mass terms descend from cubic or higher order superpotential terms. Large mass terms will only result when a singlet develops a sufficiently large vev. As will be clear in all of the models considered here, such singlets are charged under additional gauged symmetries. In this case, such mass terms may not be sufficiently large to avoid spoiling gauge coupling unification. For these reasons, we shall always require that the zero mode content of the low energy theory contains no vector-like pairs of fields in exotic representations of G_{std}.

This constraint imposes important restrictions on admissible gauge bundle configurations which can break the bulk gauge group G_S to G_{std}. In particular, when $G_S = \text{SU}(5)$, we show that the gauge bundle configurations with no exotica are in one to one correspondence with the roots of an exceptional Lie algebra corresponding to the del Pezzo surface in question. Moreover, when $G_S = \text{SO}(10)$, we present a no go theorem which shows that direct breaking of G_S to G_{std} via internal fluxes always produces exotica in the low energy theory.

10.1 Fractional line bundles

In this section we determine which internal fluxes can break the GUT group and simultaneously do not generate any extraneous zero modes in the low energy spectrum. In fact, a cursory analysis would incorrectly suggest that such states are unavoidable. For example, the decomposition of the adjoint representation of SU(5) decomposes under G_{std} as:

$$\text{SU}(5) \supset \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \tag{10.1}$$

$$24 \rightarrow (1,1)_0 + (8,1)_0 + (1,3)_0 + (3,\overline{2})_{-5} + (\overline{3},2)_5. \tag{10.2}$$

We note that no fields of the MSSM transform in the representation $(3,\overline{2})_{-5}$ or $(\overline{3},2)_5$. Letting L denote the supersymmetric line bundle associated with this breaking pattern, the bulk zero mode content therefore descends to:

$$(3,\overline{2})_{-5} \in H^0_{\overline{\partial}}(S,L^5)^* \oplus H^1_{\overline{\partial}}(S,L^{-5}) \oplus H^2_{\overline{\partial}}(S,L^5)^* \tag{10.3}$$

$$(\overline{3},2)_5 \in H^0_{\overline{\partial}}(S,L^{-5})^* \oplus H^1_{\overline{\partial}}(S,L^5) \oplus H^2_{\overline{\partial}}(S,L^{-5})^*. \tag{10.4}$$

Mathematically, the collection of admissible line bundles are those which have vanishing cohomology group. As explained in section 8, when S is a del Pezzo N surface, such line bundles are in one to one correspondence with the roots of the Lie algebra E_N, with a similar result for $\mathbb{P}^1 \times \mathbb{P}^1$. By definition, a root α satisfies the condition that $n\alpha$ is also a root only when $n = \pm 1$. It now follows that if L is a line bundle, L^5 cannot correspond to a root of the Lie algebra E_N. Said differently, the integral quantization of fluxes in the bulk theory would appear to present a general obstruction towards realizing the spectrum of the MSSM without any additional bulk matter with exotic $U(1)_Y$ charges.

We now argue that so long as all fields transform in mathematically well-defined line bundles, fractional powers of line bundles also define consistent gauge field configurations for the bulk theory. To establish this, first recall that when all fields of a theory with gauge group $SU(N)$ transform in the adjoint representation, all observables are invariant under $SU(N)$ modulo the center. Hence, the actual gauge group of the theory is $SU(N)/\mathbb{Z}_N$ so that the flux quantization condition allows gauge field configurations with $1/N$ fractional flux units [66]. In the presence of quark fields charged in the fundamental of $SU(N)$, we note that the gauge group is indeed $SU(N)$ rather than $SU(N)/\mathbb{Z}_N$.

In the present class of models, a similar fractional quantization condition holds because all of the resulting gauge groups descend from an E_8 gauge group. Indeed, recall that the E_N groups canonically embed in E_8 as:

$$\frac{E_N \times SU(K)}{\mathbb{Z}_K} \subset E_8 \tag{10.5}$$

where $N + K = 9$. This result can be established as follows. Decomposing the adjoint representations of E_N and $SU(K)$ to $E_{N-1} \times U(1)$ and $SU(K-1) \times U(1)$, we find that the resulting representations all have charge 0 or $\pm K$. As two examples, consider the decomposition of the adjoint representations of the algebras E_6 and $E_5 = SO(10)$:

$$E_6 \supset SO(10) \times U(1) \tag{10.6}$$

$$78 \to 1_0 + 45_0 + 16_{-3} + \overline{16}_3 \tag{10.7}$$

$$E_5 \supset SU(5) \times U(1) \tag{10.8}$$

$$45 \to 1_0 + 24_0 + 10_4 + \overline{10}_{-4}. \tag{10.9}$$

Returning to the weight space decomposition of the charged representations, it follows that the relative normalization of the matrices which generate the Cartan subalgebras of E_N and E_8 differ by $1/K$. Exponentiating these matrices, we arrive at the desired condition in the corresponding subgroups.

This fractional quantization condition demonstrates that in the above example, we may treat L^5 as a line bundle, with L a "fractional power" of a line bundle. Moreover, fields localized on a matter curve Σ transform as sections of $K_{\Sigma}^{1/2} \otimes L_{\Sigma}^{a} \otimes L_{\Sigma}^{\prime b}$ for integers a and b, where L_{Σ} and L'_{Σ} respectively denote the restriction of potentially fractional line bundles on S and S'. Indeed, the common identification of the centers of the gauge groups in (10.5) illustrates that although the individual restrictions of L and L' to Σ may correspond to ill-defined line bundles, their tensor product may still determine a mathematically well-defined line bundle. We therefore conclude that so long as the resulting fields all transform in well-defined bundles, the corresponding fractional line bundles are physically well-defined.

10.2 A no go theorem for $G_S = \mathrm{SO}(10)$

The analysis of the previous subsection establishes that when $G_S = \mathrm{SU}(5)$, there are no exotic bulk zero modes if and only if the gauge bundle corresponds to a fractional line bundle of the form $\mathcal{O}_S(\alpha)^{1/5}$ where α corresponds to a root associated with an element of $H_2(S, \mathbb{Z})$. In this section we show that when $G_S = \mathrm{SO}(10)$, direct breaking to G_{std} via fluxes always results in exotica in the low energy spectrum.

To establish this result, we note that the classification of appendix C shows that the only instanton configurations which break SO(10) to G_{std} take values in the subgroup $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ so that the commutant subgroup in SO(10) is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$. With respect to this decomposition, the adjoint, spinor and vector representations of SO(10) decompose as:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \times \mathrm{U}(1)_2 \supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \tag{10.10}$$

$$45 \rightarrow (1,1)_{0,0} + (1,1)_{0,0} + (8,1)_{0,0} + (1,3)_{0,0} \tag{10.11}$$

$$+ (3,\overline{2})_{-5,0} + (\overline{3},2)_{5,0} + (3,2)_{1,4} + (\overline{3},\overline{2})_{-1,-4} \tag{10.12}$$

$$+ (\overline{3},1)_{-4,4} + (3,1)_{4,-4} + (1,1)_{6,4} + (1,1)_{-6,-4} \tag{10.13}$$

$$16 \rightarrow (1,1)_{0,-5} + (\overline{3},1)_{2,3} + (1,\overline{2})_{-3,3} \tag{10.14}$$

$$+ (1,1)_{6,-1} + (3,2)_{1,-1} + (\overline{3},1)_{-4,-1} \tag{10.15}$$

$$10 \rightarrow (3,1)_{-2,2} + (1,2)_{3,2} + (\overline{3},1)_{2,-2} + (1,\overline{2})_{-3,-2}. \tag{10.16}$$

In the MSSM, fields charged under the subgroup $\mathrm{SU}(3) \times \mathrm{SU}(2)$ transform in the representations $(3,2)$, $(1,2)$ and $(\overline{3},1)$. Returning to the decomposition of the 45, we conclude that the low energy spectrum must contain no fields transforming in the $(\overline{3},2)_{5,0}$, $(\overline{3},\overline{2})_{-1,-4}$ or $(3,1)_{4,-4}$.

In F-theory, all of the matter content of the MSSM descend from the 45, 16, $\overline{16}$ or 10 of SO(10). As reviewed in appendix C, there are precisely two linear combinations of $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ which can correspond to $\mathrm{U}(1)_Y$ in the Standard Model:

$$\mathrm{U}(1)_Y = \mathrm{U}(1)_1 \tag{10.17}$$

$$\mathrm{U}(1)_Y = -\frac{1}{5}\mathrm{U}(1)_1 - \frac{6}{5}\mathrm{U}(1)_2. \tag{10.18}$$

While the first case corresponds to embedding hypercharge in the usual way inside of the SU(5) factor, the second possibility corresponds to a "flipped" embedding of hypercharge [67].

First suppose that $\mathrm{U}(1)_Y$ is given by equation (10.17). Letting $A \equiv L_1^5$ and $B \equiv L_1^{-1} \otimes L_2^{-4}$, the condition that the zero mode content must contain no exotic matter requires that the following cohomology groups must vanish:

$$(\overline{3},2)_{5,0} \in H^1_{\overline{\partial}}(S, A) = 0 \tag{10.19}$$

$$(3,\overline{2})_{-5,0} \in H^1_{\overline{\partial}}(S, A^{-1}) = 0 \tag{10.20}$$

$$(\overline{3},\overline{2})_{-1,-4} \in H^1_{\overline{\partial}}(S, B) = 0 \tag{10.21}$$

$$(3,1)_{4,-4} \in H^1_{\overline{\partial}}(S, A \otimes B) = 0 \tag{10.22}$$

$$(1,1)_{-6,-4} \in H^1_{\overline{\partial}}(S, A^{-1} \otimes B) = 0. \tag{10.23}$$

For a supersymmetric configuration, it follows from the vanishing theorem of [15] that the cohomology groups $H^0_{\overline{\partial}}$ and $H^2_{\overline{\partial}}$ vanish for all of the above line bundles. The cohomology group $H^1_{\overline{\partial}}$ therefore vanishes when the index of each line bundle vanishes. Equations (10.19) and (10.20) imply:

$$0 = \chi(S, A) + \chi(S, A^{-1}) = 2 + c_1(A) \cdot c_1(A). \tag{10.24}$$

On the other hand, equations (10.21)–(10.23) imply:

$$0 = \chi(S, A \otimes B) + \chi(S, A^{-1} \otimes B) - 2\chi(S, B) = c_1(A) \cdot c_1(A) \tag{10.25}$$

which contradicts equation (10.24). The resulting low energy spectrum will therefore always contain some exotic matter.

Next consider the flipped embedding of $\mathrm{U}(1)_Y$ given by equation (10.18). With notation as above, the condition that the zero mode content must contain no exotic matter now requires that the following cohomology groups vanish:

$$(\overline{3}, \overline{2})_{-1,-4} \in H^1_{\overline{\partial}}(S, B) = 0 \tag{10.26}$$
$$(3, 2)_{1,4} \in H^1_{\overline{\partial}}(S, B^{-1}) = 0 \tag{10.27}$$
$$(\overline{3}, 2)_{5,0} \in H^1_{\overline{\partial}}(S, A) = 0 \tag{10.28}$$
$$(3, 1)_{4,-4} \in H^1_{\overline{\partial}}(S, A \otimes B) = 0 \tag{10.29}$$
$$(1, 1)_{6,4} \in H^1_{\overline{\partial}}(S, A \otimes B^{-1}) = 0. \tag{10.30}$$

These conditions are the same as those of equations (10.19)–(10.23) with the roles of A and B interchanged. We therefore conclude that in all cases, the resulting spectrum will contain exotic matter.

More generally, we note that the classification of possible breaking patterns provided in appendix C requires at least one U(1) factor. When G_S has rank five or more, direct breaking to G_{std} therefore requires the instanton configuration to take values in a subgroup of G_S with rank at least two. We note that while only abelian instanton configurations are available for rank four and five bulk gauge groups, it is in principle possible that an SU(2) valued instanton could partially break the bulk gauge group when $G_S = E_6$. However, decomposing the adjoint representation to G_{std}, the number of different exotic representations appears to always be greater than the rank of the subgroup in which the instanton takes values. The requirement that so many different cohomology groups must simultaneously vanish is then an over-constrained problem so that in such cases exotics are unavoidable.

10.3 MSSM spectrum

In this section we explain how to obtain the exact spectrum of the MSSM when S is a del Pezzo surface. As explained in subsection 10.2, direct breaking via internal fluxes will generate exotics when the bulk gauge group is not SU(5). Restricting to the case

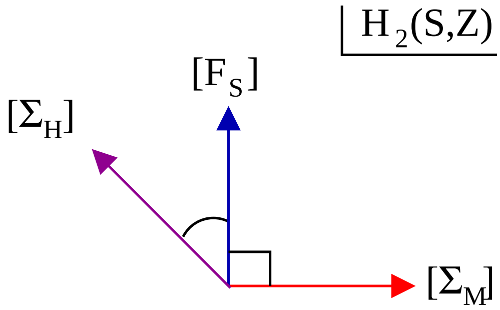

Figure 4: Letting $[F_S]$ denote the two-cycle in $H_2(S,\mathbb{Z})$ which is Poincaré dual to the background hypercharge flux $\langle F_S \rangle$, there is a natural distinction between the class of the Higgs curve $[\Sigma_H]$ and the class of the chiral matter curves $[\Sigma_M]$. Indeed, while the net flux on Σ_M must vanish to preserve a full GUT multiplet, the gauge field configuration must restrict non-trivially on the Higgs curves in order to solve the doublet triplet splitting problem. When the net flux on the Higgs curve is not zero, this corresponds to the condition that $[\Sigma_M]$ and $[F_S]$ are orthogonal while $[\Sigma_H]$ and $[F_S]$ are not.

$G_S = \mathrm{SU}(5)$, the only candidate bundles which will not generate exotic bulk zero modes are in one to correspondence with the roots of an exceptional Lie algebra. In this case, all of the matter content of the MSSM must localize on matter curves.

Individual components of a GUT multiplet will interact differently with the internal hypercharge flux. In keeping with our general philosophy, we require that a complete GUT multiplet must localize on a given matter curve so that on such curves, the net hypercharge flux must vanish. Otherwise, a different index will determine the number of zero modes coming from each component of a complete GUT multiplet. On the other hand, the gauge field must restrict non-trivially on the Higgs curves in order to solve the doublet-triplet splitting problem. See figure 4 for a depiction of how the corresponding elements in $H_2(S,\mathbb{Z})$ intersect.

In order to achieve a chiral matter spectrum in four dimensions, the net flux on the matter curve cannot vanish. As an example, consider a six-dimensional hypermultiplet in the 5_1 of $G_S \times G_{S'} = \mathrm{SU}(5) \times \mathrm{U}(1)$ which localizes on an exceptional curve Σ with homology class E_1. The overall normalization of the U(1) charge is not particularly important because we shall consider vacua with fractional line bundles. When $L = \mathcal{O}_S(E_2 - E_3)^{1/5}$ the restriction of L to Σ is trivial. Letting $L^{'}$ denote the supersymmetric line bundle on the seven-brane which intersects the GUT model seven-brane along Σ, the restriction of L' to Σ must be non-trivial in order to achieve a chiral matter spectrum. For example, when $L'_\Sigma = \mathcal{O}_\Sigma(-3)$, the zero mode content is:

$$0 \times 5 \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes \mathcal{O}_\Sigma(-3)) = 0 \tag{10.31}$$

$$3 \times \overline{5} \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes \mathcal{O}_\Sigma(3)) = H^0_{\overline{\partial}}(\Sigma, \mathcal{O}_\Sigma(2)) \tag{10.32}$$

where we have also indicated the multiplicity of the zero modes. Similar considerations apply for other GUT multiplets.

On the other hand, the Higgs fields of the MSSM do not fill out complete GUT multiplets at low energies. In this case, the net hypercharge flux piercing this matter curve must be non-zero. More precisely, recall that for minimal supersymmetric SU(5) GUT models, the Higgs up and down fields respectively descend from the 5 and $\overline{5}$ of SU(5), where the 5 decomposes to $(3,1)_{-2}+(1,2)_3$. Letting L_Σ denote the restriction of the bulk gauge bundle L to the matter curve Σ with similar notation for L'_Σ, we note that the zero mode content is determined by the cohomology groups:

$$(1,2)_3 \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes L_\Sigma^3 \otimes L_\Sigma'^n) \tag{10.33}$$

$$(3,1)_{-2} \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes L_\Sigma^{-2} \otimes L_\Sigma'^n) \tag{10.34}$$

where n is an integer associated with the U(1) charge associated with the brane wrapping S'. Mathematically, we wish to find line bundles such that $K_\Sigma^{1/2} \otimes L_\Sigma^3 \otimes L_\Sigma'^n$ has non-vanishing cohomology whereas $K_\Sigma^{1/2} \otimes L_\Sigma^{-2} \otimes L_\Sigma'^n$ has trivial cohomology. A necessary condition for $K_\Sigma^{1/2} \otimes L_\Sigma^{-2} \otimes L_\Sigma'^n$ to have trivial cohomology is that the degree of the line bundle $L_\Sigma^{-2} \otimes L_\Sigma'^n$ must vanish. As a brief aside, we recall the well-known fact that degree zero line bundles are in one to one correspondence with points on the Jacobian of the curve.

As an example, consider a genus one matter curve where the line bundles L_Σ and L'_Σ are given by:

$$L_\Sigma = \mathcal{O}_\Sigma(-np_1 + np_2) \tag{10.35}$$

$$L'_\Sigma = \mathcal{O}_\Sigma(3p_1 - 3p_2) \tag{10.36}$$

where p_1 and p_2 denote distinct degree one divisors on Σ which are not linearly equivalent. Because these divisors are not linearly equivalent, the divisor $p_1 - p_2$ is not effective.[16] Assuming that $K_\Sigma^{1/2}$ is trivial, we have:

$$(1,2)_3 \in H^0_{\overline{\partial}}(\Sigma, \mathcal{O}_\Sigma(0)) \simeq \mathbb{C} \tag{10.37}$$

$$(3,1)_{-2} \in H^0_{\overline{\partial}}(\Sigma, \mathcal{O}_\Sigma(5n(p_1 - p_2)) = 0 \tag{10.38}$$

since the divisor $p_1 - p_2$ is not effective. In this case, we achieve a vector-like pair of Higgs up/down fields on the curve Σ.

Now, there is no reason that the Higgs up and down fields must localize on the same matter curve. In a certain sense, the above implementation of doublet triplet splitting is somewhat artificial precisely because the distinguishing feature of the Higgs curve is that a non-trivial flux is present. With this in mind, it seems far more natural to consider line bundles which have non-trivial degree on the Higgs curves. In this case, a given Higgs curve will automatically contain more Higgs up than Higgs down fields.

To give an explicit example of this type, consider a six-dimensional hypermultiplet in the 5 of SU(5) localized on a genus zero curve Σ. In this case, the zero mode content is determined by the cohomology groups:

$$(1,2)_3 \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes L_\Sigma^3 \otimes L'_\Sigma) \tag{10.39}$$

$$(1,\overline{2})_{-3} \in H^0_{\overline{\partial}}(\Sigma, K_\Sigma^{1/2} \otimes L_\Sigma^{-3} \otimes L_\Sigma'^{-1}) \tag{10.40}$$

[16] More generally, recall that on a general genus g Riemann surface, a divisor D with degree $\geq g$ is linearly equivalent to an effective divisor [68]. This imposes a non-trivial constraint on the ways in which doublet-triplet splitting can arise for a general matter curve.

$$(3,1)_{-2} \in H^0_{\overline{\partial}}(\Sigma, K_{\Sigma}^{1/2} \otimes L_{\Sigma}^{-2} \otimes L'_{\Sigma}) \tag{10.41}$$

$$(\overline{3},1)_{2} \in H^0_{\overline{\partial}}(\Sigma, K_{\Sigma}^{1/2} \otimes L_{\Sigma}^{2} \otimes L'^{-1}_{\Sigma}). \tag{10.42}$$

The zero mode content on Σ yields precisely one Higgs up field for fractional line bundle assignments:

$$L_{\Sigma} = \mathcal{O}_{\Sigma}(1)^{1/5} \text{ and } L'_{\Sigma} = \mathcal{O}_{\Sigma}(1)^{2/5}. \tag{10.43}$$

Similarly, a single Higgs down field can also localize on another matter curve.

It is also possible to localize a single Higgs up field on a higher genus matter curve. For example, with notation as above, when Σ is a genus one curve, the fractional line bundle assignments:

$$L_{\Sigma} = \mathcal{O}_{\Sigma}(p_1)^{1/5} \otimes \mathcal{O}_{\Sigma}(p_1-p_2)^{1/5} \text{ and } L'_{\Sigma} = \mathcal{O}_{\Sigma}(p_1)^{2/5} \otimes \mathcal{O}_{\Sigma}(p_1-p_2)^{-3/5} \tag{10.44}$$

will again yield a single H_u field localized on Σ.

In fact, in section 12 we will show that in order to remain in accord with current bounds on the lifetime of the proton, the Higgs fields *must* localize on different matter curves. These matter curves may or may not intersect inside of S. When these curves do not intersect, these fields do not couple in the superpotential and the μ term is automatically zero. Moreover, when these curves do intersect, they must interact with a third gauge singlet which localizes on a curve that only intersects S at a point. In section 15 we estimate the behavior of this gauge singlet wave function near the surface S and show that this naturally yields an exponentially suppressed μ term.

10.4 Candidates for dark matter

In the MSSM with R-parity, the lightest supersymmetric partner (LSP) could be a viable dark matter candidate. In fact, in the context of a local model, it is natural to expect a large number of additional gauge degrees of freedom which only interact gravitationally with the MSSM. This appears to be an automatic feature of many consistent string compactifications which will typically contain several hidden sectors. For example, in the perturbative heterotic string, this role can be played by the hidden E_8 factor. A rough comparison of the two E_8 factors would then suggest that half of the matter content in such a model could be visible, and the other half could be dark matter. In F-theory, the analogue of the hidden E_8 factor could be the additional seven-branes which are required for the compactification to be globally consistent. For example, F-theory compactified on $K3$ corresponds to a configuration of 24 seven-branes. More generally, it would be of interest to estimate the number of seven-branes which only interact gravitationally with the MSSM. In this case, the total class of the seven-branes in a threefold base B_3 is given by $12c_1(B_3)$. Integrating this Chern class over an appropriate two-cycle would then yield a rough estimate on the amount of dark matter from seven-branes. It is also in principle possible that the total number of three-branes in the compactification could also contribute to the dark matter content of the model. In the absence of fluxes, the total number of three-branes is given by $\chi(CY_4)/24$. We note that in order for the Calabi-Yau fourfold to be elliptically fibered, the threefold base B_3 must be a Fano variety. For example, $B_3 = \mathbb{P}^3$, gives 48 seven-branes.

Note that the GUT group involves a bound state of $O(10)$ such seven-branes. We find it quite amusing that this is in rough agreement with the observed ratio between visible and dark matter in our Universe! Of course, this depends on the relative masses for the various visible and hidden fields. There is a finite list of such manifolds [69], and it would therefore be of interest to compare the relative number of three-branes and seven-branes in such compactifications.

11. Geometry and matter parity

From a phenomenological viewpoint, matter parity provides a simple way to forbid renormalizable terms in the four-dimensional superpotential which can potentially induce proton decay. It also naturally leads to an LSP which could potentially be a dark matter candidate. In a Lorentz invariant theory, this is equivalent to assigning an appropriate R-parity to the individual components of a superfield. Indeed, the essential point is that this discrete symmetry distinguishes the Higgs superfields from all of the other chiral superfields of the MSSM. In this section we argue that the presence of such a $\mathbb{Z}_2$ symmetry is quite natural from the perspective of F-theory.

As explained in subsection 10.3, the Higgs fields localize on matter curves pierced by a net amount of internal hypercharge flux while the chiral matter localizes on curves where the net hypercharge flux is trivial. This is a discrete choice which naturally distinguishes the Higgs superfields from the rest of the chiral superfields of the MSSM.

From a more global perspective, these fluxes correspond to the localization of four-form G-flux in the compactification. If the Calabi-Yau fourfold admits a geometric $\mathbb{Z}_2$ symmetry, then these fluxes will decompose into even and odd elements of $H^4(CY_4, \mathbb{Z})$ which we denote by $H^4(CY_4, \mathbb{Z})_+$ and $H^4(CY_4, \mathbb{Z})_-$. If this symmetry is well-defined, it follows that on a given seven-brane, the corresponding line bundles must have a definite parity under this choice of sign. For example, the parity of the line bundle on the S brane can be even while the parity of the line bundles on the other branes may have other parities.

It now follows that the net flux on a matter curve can only be non-zero when the flux and matter curve have the same parity. Indeed, letting $F^{\pm}$ denote a flux with parity ± 1 with similar notation for matter curves $\Sigma^{\pm}$, the unbroken $\mathbb{Z}_2$ symmetry implies:

$$\int_{\Sigma^-} F^+_{\Sigma^-} = \int_S F^+ \wedge PD(\Sigma^-) = -\int_S F^+ \wedge PD(\Sigma^-) = 0 \tag{11.1}$$

$$\int_{\Sigma^+} F^-_{\Sigma^+} = \int_S F^- \wedge PD(\Sigma^+) = -\int_S F^- \wedge PD(\Sigma^+) = 0 \tag{11.2}$$

where $PD(\Sigma^{\pm}) \in H^2(S, \mathbb{Z})$ denotes the Poincaré dual element of $[\Sigma^{\pm}] \in H_2(S, \mathbb{Z})$. In other words, when the integral of the flux over a curve does not vanish, the flux and curve have the *same* parity.

In order for this group action to remain well-defined on the matter curves, the internal wave functions which are sections of appropriate bundles must also have a definite sign under the group action. First consider the parity of the Higgs fields. These wave functions are defined as sections of line bundles which depend non-trivially on the restriction of line bundles from both S as well as other transversely intersecting seven-branes. We therefore conclude that both fluxes must have the same parity. In particular, we conclude if the parity of the bulk gauge field is even, then the Higgs fields will also have even parity.

Next consider the parity of the remaining matter fields. Here it is essential that the net flux contribution from S is trivial on all such matter curves. In particular, if the gauge bundle from the transversely intersecting seven-brane is odd under matter parity, then the corresponding sections on each matter curve will also be odd under the $\mathbb{Z}_2$ action on the Calabi-Yau fourfold. Hence, we obtain on rather general grounds a geometric version of matter parity.

12. Proton decay and doublet-triplet splitting

As argued in subsection 10.3, there exist vacua which yield the exact spectrum of the MSSM for an appropriate choice of flux in a local intersecting seven-brane configuration. In particular, we found that the Higgs triplets can typically be removed from the low energy spectrum. While this mechanism provides a natural way to achieve the correct zero mode spectrum in the Higgs sector, when the Higgs up and down fields localize on the same matter curve, the higher Kaluza-Klein modes of the corresponding six-dimensional fields will generate higher order superpotential terms of the form $QQQL/M_{KK}$ with order one coefficients. While here we have presented the operator in terms of the Kaluza-Klein mass scale M_{KK}, for minimal SU(5) GUT models, we can reliably approximate M_{KK} by M_{GUT}.[17] If present, such operators can significantly shorten the lifetime of the proton.

We now explain how such terms could potentially be generated in our class of models. When all Yukawa couplings to the Higgs triplets are order one parameters, the superpotential terms:

$$W_{\mathrm{GUT}} = QQT_u + QLT_d + M_{KK}T_uT_d \tag{12.1}$$

will give a large mass to the Higgs triplets T_uT_d of order M_{KK}. Integrating out T_u and T_d, the coefficient of the operator $QQQL/M_{KK}$ would then be too large to satisfy present constraints. In fact, the geometry of the matter curves indicates precisely when we can expect such terms to be generated. The tree level diagram which generates the offending operator is given by drawing the intersection locus of the matter curves and interpreting each matter curve as a leg of the corresponding Feynman diagram. See figure 5 for a depiction of how the geometry of the matter curves quite literally translates into a statement about diagrams in the low energy theory.

While it is in principle possible to suppress the value of this coefficient by incorporating flavor symmetries, in the context of four dimensional supersymmetric GUT models, this problem can be avoided by having T_u and T_d develop masses by pairing with additional

[17] When we present some examples of four-dimensional flipped SU(5) models which descend from an eight-dimensional SO(10) model, there can be a small discrepancy between the four-dimensional GUT scale M_{GUT} and M_{KK}.

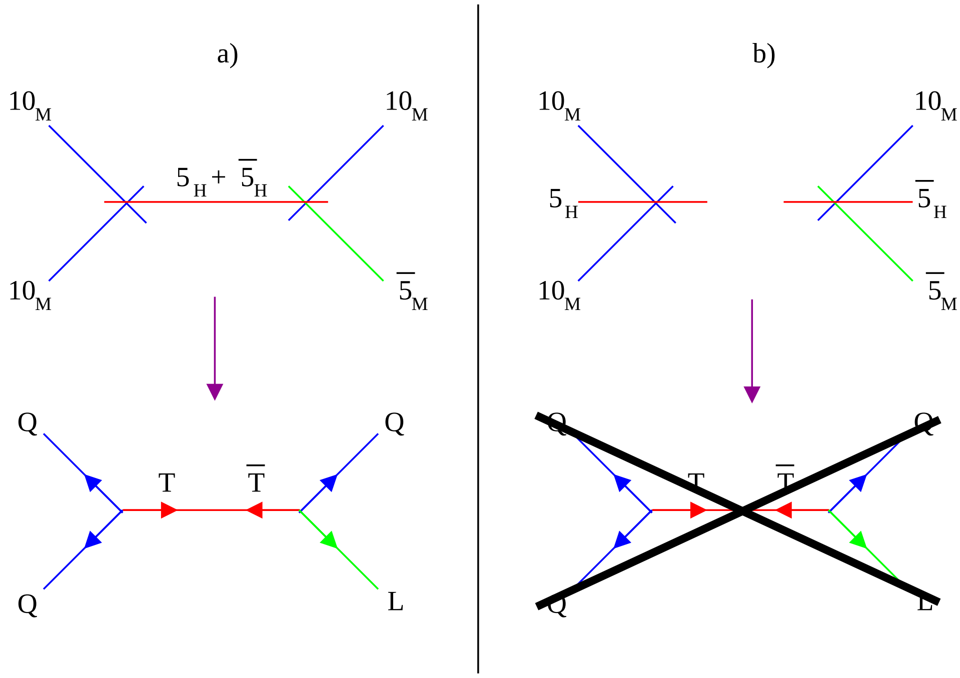

Figure 5: Depiction of how the geometry of matter curves directly translates into amplitudes in the low energy theory. In case a), the Higgs up and down fields localize on the same matter curve. The resulting field theory diagram which generates the operator $QQQL$ is given by interpreting each matter curve as the leg of a Feynman diagram. In case b), the Higgs up and down fields localize on distinct matter curves. In this case, the Feynman diagram involving the exchange of massive Higgs triplets is unavailable.

heavy triplet states T_u' and T_d' so that the superpotential instead takes the form:[18]

$$W_{\rm GUT} = QQT_u + QLT_d + MT_uT_d' + MT_dT_u' \tag{12.2}$$

which does not generate the offending dimension five operator from integrating out massive fields at tree level. Note that this occurs automatically when the Higgs up and down fields localize on distinct matter curves.

In compactifications of the heterotic string on Calabi-Yau threefolds, the Higgs triplet is typically projected out of the low energy spectrum by discrete Wilson lines. In general, it is not clear to us whether this sufficiently suppresses proton decay. Indeed, while the Higgs triplet zero mode may be absent from the spectrum, there is an entire tower of Kaluza Klein modes which must also be considered. If any of these modes contribute an interaction term of the form given by equation (12.1), the coefficient of the offending dimension five operator may still be too large to remain in accord with observation.

To summarize, we have seen that the proton decays too rapidly when the Higgs up and down fields localize on the same matter curve. As a necessary first step, we have shown that when these Higgs fields localize on distinct matter curves, integrating out the higher

[18] We thank S. Raby for emphasizing this point to us.

Kaluza-Klein modes for the Higgs fields does not generate the offending baryon number violating term $QQQL$. Even so, it is still in principle possible that some exotic process could generate the operator $QQQL$. In fact, placing the Higgs fields on different matter curves automatically equips them with additional global symmetries in the low energy effective theory. As we now explain, these symmetries significantly extend the lifetime of the proton.

13. Extra U(1)'s and higher dimension operators

In section 12 we have shown that the dimension five operators responsible for proton decay are naturally suppressed when the Higgs up and Higgs down fields localize on different matter curves. In this section we explain from a different perspective why this suppression occurs and also discuss on more general grounds when we expect other higher dimensional operators to suffer a similar fate.

Imposing additional global symmetries provides one common way to suppress undesirable interaction terms in field theory. Indeed, so long as the global symmetry remains unbroken, all of the higher order terms of the effective superpotential will also respect this symmetry. In F-theory, these U(1) factors occur automatically because the breaking direction in the Cartan subalgebra of a given singularity determines the location of the matter curves in the geometry. Matter localizes on the curve precisely when it is charged under the appropriate subgroup. While this generically allows local triple intersections of matter curves to take place, all of the fields of the MSSM will therefore be charged under additional U(1) factors. These extra U(1)'s can therefore naturally suppress higher dimension operators. When two curves do not intersect inside of S, fields localized on each curve will be charged under distinct U(1) groups. This can forbid cubic interaction terms as well as many higher order contributions to the effective superpotential. It would be very interesting to determine the precise mapping between topological properties of intersecting curves and the associated U(1) fields.

From a bottom up perspective, the fields of the MSSM contain various accidental symmetries. Assuming generic values of the Yukawa couplings and that the μ term originates from the vev of a gauge singlet, the classical action is invariant under four U(1) symmetries. These can be identified with $\mathrm{U}(1)_Y$ hypercharge, $\mathrm{U}(1)_B$ baryon number, $\mathrm{U}(1)_{B-L}$ baryon minus lepton number and a $\mathrm{U}(1)_{PQ}$ Peccei-Quinn symmetry. Of these four possibilities, only $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_{B-L}$ are potentially non-anomalous.

In a quantum theory of gravity, any global symmetry must be promoted to a gauge symmetry. One potential worry is that because the fields of the MSSM are naturally charged under these U(1)'s, the presence of these gauge bosons could lead to conflict with experiment. While these U(1)'s will typically be anomalous and therefore lift from the low energy spectrum, it is interesting to ask whether a massless U(1) of this type is already ruled out by experiment. This is not very promising because current constraints from fifth force experiments have set a strong limit on the gauge coupling of extra massless U(1) gauge bosons:

$$g_{\mathrm{extra}} \lesssim \frac{m_n}{M_{\mathrm{pl}}} \sim 10^{-19} \tag{13.1}$$

where m_n denotes the mass of the neutron. In the absence of a natural explanation for why the gauge coupling would be so weak for such couplings, this appears quite fine-tuned. In fact, such a small value is already in conflict with the conjecture that gravity is the weakest force [70]. See [71, 72] for further discussion on extra massless U(1) gauge bosons.

The analogue of equation (4.8) for $\alpha_{\rm extra} = g^2_{\rm extra}/4\pi$ is of the form:

$$\alpha^{-1}_{\rm extra} = M^4_* Vol(S_{\rm extra}) \sim M^4_* R^2_\perp R^2_S \tag{13.2}$$

where as before, $R_\perp$ denotes the length scale associated with the direction normal to the surface S. In tandem with equation (4.8) this implies:

$$\alpha_{\rm extra} \sim \alpha_{\rm GUT} \frac{R^2_S}{R^2_\perp} = \alpha_{\rm GUT} \times \varepsilon^\gamma \sim 7 \times 10^{-3\pm 0.5}. \tag{13.3}$$

Based on the above estimate, we conclude that all additional U(1) gauge bosons must develop a sufficiently large mass in order to lift from the low energy spectrum. In fact, our expectation is that this only imposes a mild constraint on the compactification. When the U(1) symmetry is anomalous, the Green-Schwarz mechanism will generate a string scale mass for the gauge boson. Even when the U(1) symmetry is non-anomalous, the gauge boson can still develop a large mass. Indeed, although the analysis of section 9 shows that four-dimensional U(1) gauge bosons can remain massless in the presence of internal fluxes, it also establishes sufficient conditions for such bosons to develop a large mass on the order of $R^{-1}_\perp$. In either case, we therefore expect that it is always possible for all extraneous U(1) gauge bosons to develop a suitably large mass. In the low energy effective theory, some imprint of the gauge symmetry will remain as an approximate global symmetry in the low energy effective theory. These global symmetries can be violated by non-perturbative contributions to the superpotential from Euclidean branes wrapping the various Kähler surfaces of the compactification. Such contributions are naturally suppressed by an exponential factor of the form $\exp(-c/\alpha_{\rm extra})$ where c is an order one positive number. Similar instanton effects have been proposed as a possible solution to the cosmological constant problem [73]. Such exponentials could also provide a novel method of generating contributions to the flavor sector of the theory. We present one brief speculation along these lines in section 14. As a brief aside, recall that in section 6 we presented an example of a four-dimensional GUT model where an appropriate operator generated by non-perturbative contributions could produce an effective μ term. Indeed, when a strict decoupling limit does not exist, it is likely that non-perturbative contributions to the superpotential could play a more prominent role in the effective theory.

14. Towards realistic Yukawa couplings

Finding vacua with the correct matter spectrum of the MSSM is only the first step in constructing a semi-realistic model. In models where all chiral matter localizes on matter curves, the leading order contribution to the four-dimensional effective superpotential originates from the triple intersection of matter curves. After presenting a general analysis of how matter curves can form triple intersections in S, we show that in order to achieve one generation with mass which is hierarchically larger than the two lighter generations, some of the matter curves must self-intersect or "pinch" inside of S. See figure 6 for a depiction of a pinched curve. While a complete theory of flavor is beyond the scope of this paper, we can nevertheless provide a qualitative explanation for why the heaviest generation obeys an approximate GUT mass relation which is violated by the lighter generations. In fact, the effect we discover is *generically realized* in vacua with non-zero internal hypercharge flux because the Aharanov-Bohm effect distorts the wave functions of individual components of a GUT multiplet by different amounts. Moreover, this distortion becomes more pronounced as the mass of the generation decreases. We conclude by presenting some speculations on how more detailed properties of flavor physics could originate from a local del Pezzo model.

14.1 Criteria for triple intersections

As reviewed in section 3, cubic contributions to the superpotential of an exceptional seven-brane can originate from three sources. These correspond to interactions amongst three bulk zero modes, interactions between a single bulk zero mode and two zero modes localized on a matter curve, and interaction terms between three zero modes on matter curves. As explained in subsection 10.3, in a minimal SU(5) GUT all of the field content of the MSSM localizes on curves. Thus, the leading order contribution to the effective superpotential comes from the triple intersection of matter curves.

Locally, the triple intersection of matter curves in S occurs when the bulk singularity type G_S undergoes an at least twofold enhancement to a singularity of type $G_p \supset G_S \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$. Following the general philosophy of [44], we note that matter localized along curves in S is charged under the corresponding $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ subgroup. Indeed, letting t_1 and t_2 denote the local deformation parameters associated with the two U(1) factors, this curve is locally described by an equation of the form:

$$at_1 + bt_2 = 0. \tag{14.1}$$

In the above, the constants a and b are determined by the decomposition of the adjoint representation of G_p to $G_S \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ so that the appropriate irreducible representation of G_S has $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ charge (a, b). This is simply the statement that because the Cartan subgroup is visible to the geometry, this local enhancement in singularity type has been Higgsed in the bulk to G_S.

The triple intersection of three curves Σ_1, Σ_2 and Σ_3 requires that the intersection product of the corresponding homology classes satisfies $[\Sigma_i] \cdot [\Sigma_j] > 0$ for $i \neq j$. Even so, generic curves representing each class which all intersect pairwise will not form a triple intersection in S. However, in certain cases there exist representative holomorphic curves of each homology class which can form a triple intersection inside of S. For this to occur, it must be possible to deform the point of intersection of one pair of curves to coincide with the point of intersection of another pair. In other words, the normal bundle $N_{\Sigma/S}$ of one of the curves must possess at least one global section. Although from the perspective of the surface S this may appear to be a somewhat non-generic situation, we note that in F-theory such points of triple intersection occur automatically. Indeed, as explained in [15], this follows from the fact that in F-theory, rank two enhancements in the singularity type will generically occur at points in S. The claim now follows from group theoretic considerations.

At a pragmatic level, given curves $\Sigma_i = (f_i = 0)$, it is possible to engineer a triple intersection by requiring that one of the f_i is a linear combination of the other two f_i's in the ring of sections on S. Assuming without loss of generality that f_3 is given by a linear combination of f_1 and f_2, this can be written as:

$$f_3 = \alpha_1 f_1 + \alpha_2 f_2 \tag{14.2}$$

where the α_i correspond to holomorphic sections of some line bundles on S. For example, this condition is satisfied when both $[\Sigma_3] - [\Sigma_1]$ and $[\Sigma_3] - [\Sigma_2]$ are "effective" divisors, namely divisors which correspond to holomorphic curves.

This geometric condition can be used to narrow the search for vacua which are phenomenologically viable. For example, to forbid cubic matter parity violating contributions to the superpotential, it is enough to require that the curves supporting the chiral matter of the MSSM must not form a triple intersection. On the other hand, in order to have at non-trivial interaction terms, some of the matter content of the MSSM must localize on a curve which *is not exceptional*. Indeed, three exceptional curves cannot triple intersect in S. This follows from the fact that the normal bundle of a curve in S has degree $[\Sigma] \cdot [\Sigma]$ which equals -1 for an exceptional curve. Because $H^0_{\overline{\partial}}(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, none of the pairwise intersection points in such a configuration can be deformed to a point of triple intersection.

14.2 Textures

At zeroth order, it is most important to obtain a naturally heavy third generation in the quark sector. Indeed, the mass of the top quark is roughly 170 GeV, which is significantly higher than the next heaviest up type quark. This requires that the corresponding Yukawa coupling must be sufficiently large. In a suitable basis, we therefore require that the up-type Yukawa couplings are of the form:

$$\lambda^u \sim \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & 1 \end{bmatrix} \tag{14.3}$$

where the ε's are all parametrically smaller than 1.

When all of the cubic terms of the superpotential originate from the triple intersection of matter curves in S, there is additional structure in the form of the Yukawa couplings. First consider the Yukawa couplings for fields charged in the 10 of SU(5). In this case, the interaction terms:

$$W \supset \lambda^u_{ij} 5_H \times 10^{(i)}_M \times 10^{(j)}_M \tag{14.4}$$

are non-zero whenever the curves defined by Σ_H, Σ_i and Σ_j form a triple intersection. When none of the Σ_i self-intersect, or "pinch", it follows that the general form of λ^u_{ij} is:

$$\lambda^u_{ij} = \begin{bmatrix} 0 & A & B \\ A & 0 & C \\ B & C & 0 \end{bmatrix} \tag{14.5}$$

where A, B and C are constants given by evaluating wave function overlaps. We now argue that this matrix cannot yield one generation which is hierarchically heavier than the first two generations. In order for such a hierarchy to exist, we require that there exists a limit in the parameters A, B and C where two of the masses determined by λ^u_{ij} tend to zero while the third mass remains large.

In the limit in which one of the generations has zero mass, the determinant of the matrix λ^u_{ij} vanishes:

$$2ABC = 0 \tag{14.6}$$

so that without loss of generality, we may assume that the strictly massless limit corresponds to $A = 0$. Since the trace of λ^u is zero, we conclude that when $A = 0$, two of the eigenvalues of λ^u are equal in magnitude and have opposite sign. This implies that there does not exist a limit in which two of the generations are parametrically lighter than the third. On the contrary, this would suggest that two of the generations are significantly heavier than the lightest generation. We emphasize that this result holds independent of how the kinetic terms are normalized. This is because it is always possible to switch to a basis of fields where the kinetic terms are canonically normalized. This alters the form of λ^u by a similarity transformation and an overall rescaling. In this new basis, the determinant and trace will still vanish so that the above argument proceeds as before.

14.2.1 Self-intersecting or pinched curves

Rather than appeal to non-perturbative effects, we note that such a hierarchy can easily be achieved provided the Yukawa matrix possesses at least one non-zero diagonal element. Geometrically, this requires that one of the matter curves must pinch off so that globally, the curve intersects itself inside of S. We caution that this notion of self-intersection is somewhat stronger than what is usual meant by self-intersection at the level of homology. At the level of homology, a class is typically said to self intersect when two distinct representatives of a given homology class intersect inside of S. See figure 6 for a depiction of how a curve can self-intersect by pinching off inside of S.

We now extend the analysis of [15] for smooth matter curves to the present case of interest where the curve may pinch off, or self-intersect. Before describing the case of self intersection, let us recall what happens when two distinct curves intersect. In this case near the generic intersection point the two curves can be modeled by the equation:

$$z_1 z_2 = 0 \tag{14.7}$$

where $z_1 = 0$ describes one curve and $z_2 = 0$ denotes the other so that the intersection point is located at $z_1 = z_2 = 0$. By group theory considerations explained in [15], it is clear

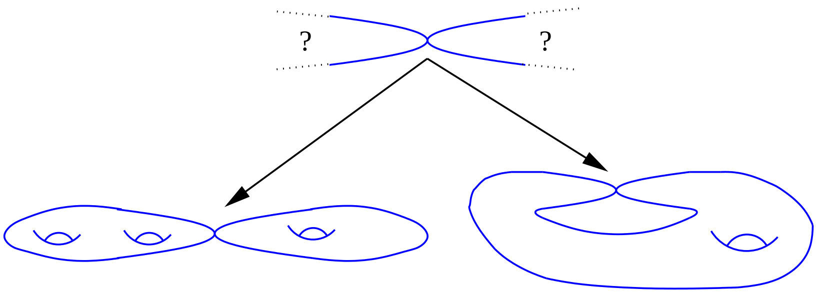

Figure 6: Depiction of how a local enhancement in the singularity type can enhance to the intersection of two distinct curves (left), or a single curve which self-intersects (right).

that a third matter curve will also pass through this point, with a local defining equation $z_3 = z_1 + z_2 = 0$. This gives rise to a Yukawa interaction of the form:

$$W \supset \phi_1 \phi_2 \phi_3 \tag{14.8}$$

where ϕ_i denotes a field associated with the local vanishing locus $z_i = 0$.

From a global perspective, this description does not specify whether ϕ_1 and ϕ_2 localize on distinct matter curves or whether they localize on the same curve. In the case where these fields localize on the same curve, the locus $z_1 = 0$ curve must connect to the $z_2 = 0$ curve in a more global description inside of S. In other words, these two loci must form a single Riemann surface. Hence, a self-intersecting curve corresponds to a genus $g+1$ curve which pinches to a genus g curve in such a way that this pinching process does not lead to two disconnected surfaces. Conversely, when this pinching process produces two disconnected curves, this describes the case where the matter curves are distinct.

To analyze the matter content localized on a self-intersecting curve, we note that the overlap of wave functions at the pinching point determines a single linear relation amongst the various zero modes of the form:

$$\alpha_i \phi^{(i)} = 0 \tag{14.9}$$

where the $\phi^{(i)}$ label the zero modes of the genus g curve obtained by pinching the associated genus $g+1$ curve. This identification reduces the value of the associated index by one.

The number of self-intersection points as well as their proximity will clearly have an impact on the properties of the Yukawa couplings in the low energy theory. To illustrate this point, it is enough to consider the up type Yukawa couplings of the minimal GUT model which descend from the cubic interaction term:

$$W \supset \lambda^u_{ij} 5_H \times 10^i_M \times 10^j_M. \tag{14.10}$$

Suppose that three generations in the 10 of SU(5) all localize on the same self-intersecting matter curve. If there is only one point of self-intersection which we denote by 0, the Yukawa matrix is given by the outer product of the wave function for the three generations:

$$\lambda^u_{ij} = \psi_H(0) \psi_i(0) \psi_j(0) \tag{14.11}$$

so that it automatically has rank one. By a suitable change of basis, the leading order behavior of the up-type Yukawa couplings is given by equation (14.3) as required in a semi-realistic model. Additional points of self-intersection will increase the rank of the up-type Yukawa coupling matrix. In this case, the relative proximity between these points of intersection as well the analogous expressions for the down-type Yukawa couplings will control the masses and mixing angles in the quark sector. It would be interesting to determine whether a hierarchical pattern of masses and mixing angles could emerge from such a treatment.

14.3 GUT mass relations

In this subsection we show that the usual GUT mass relations present in the simplest four-dimensional GUT models can be significantly distorted in the presence of an internal hypercharge flux. In the simplest four-dimensional GUT models, the masses of the up and down type quarks are determined by the superpotential terms:

$$W \supset \lambda_{ij}^{u} 5_H \times 10_M^i \times 10_M^j + \lambda_{ij}^{d} \overline{5}_H \times \overline{5}_M^i \times 10_M^j. \tag{14.12}$$

Assuming that the individual components of a GUT multiplet have the same wave function normalization, this would imply that $m_q = m_l$ for the quarks and leptons which unify in a $\overline{5}_M$ of SU(5). Evolving the values of the masses observed at low energies up to the GUT scale, it is well-known that only the third generation obeys a relation of the form $m_b \sim m_\tau$. At the level of precision we can perform here, the original analysis of mass relations in the non-supersymmetric SU(5) GUT analyzed in [74] is certainly sufficient. In this case, the actual mass relations at the GUT scale are:

$$m_b \sim m_\tau,\ m_s \sim m_\mu/3,\ m_d \sim 3m_e. \tag{14.13}$$

See [75] for an updated analysis of the various mass relations obtained by extrapolating the observed values of the masses to the GUT scale. This problem is even more pronounced for the simplest SO(10) GUTs where all interaction terms descend from the coupling $16_M \times 16_M \times 10_H$. Letting $i = 1, 2, 3$, we can parameterize the violation of the expected mass relation for each generation:

$$\delta_i = \frac{m_q^{(i)} - m_l^{(i)}}{m_q^{(i)} + m_l^{(i)}}. \tag{14.14}$$

Returning to equation (14.13), the violation of the simplest mass relation for each generation is:

$$\delta_3 = 0,\ \delta_2 \sim 50\%,\ \delta_1 \sim -50\%. \tag{14.15}$$

In purely four-dimensional GUT models, one popular way to rectify the above problems requires introducing higher-dimensional representations which couple differently to the individual components of a full GUT multiplet. It is also common to introduce adjoint-valued chiral superfields which can couple to the chiral matter of the MSSM through higher dimension operators. In both approaches, the field content necessary to avoid some of the problematic mass relations of the simplest GUT models is unavailable in a del Pezzo model!

In higher dimensional theories, additional mechanisms are potentially available. In compactifications of the heterotic string on Calabi-Yau manifolds, the particle content can organize into GUT multiplets while the wave functions corresponding to a given generation may not admit such a simple interpretation. For example, in the presence of a discrete Wilson line which breaks the GUT group to the Standard Model gauge group, individual components of a GUT multiplet may be projected out. In this way, some of the usual mass relations could become ambiguous [76]. Further, additional mixing terms between vector-like pairs of massive Kaluza-Klein modes can also obscure the meaning of simple GUT mass relations. An example of this type is discussed in [40]. Similar ideas are also quite common in orbifold GUT models. In a minimal SU(5) GUT model of the type treated here, one extreme solution would be to invoke the mechanism of doublet triplet splitting via fluxes described in section 12 so that individual components of a full GUT multiplet could localize on distinct matter curves.

While this provides one possible way to avoid incorrect mass relations amongst members of the lighter generations, we find it somewhat anti-thetical to the whole idea of grand unification that the matter content of the Standard Model neatly fits into GUT multiplets. Indeed, it would seem unfortunate to sacrifice such an aesthetic motivation for grand unification. Moreover, the usual GUT mass relation does work relatively well for the third generation. We now argue that even when a complete GUT multiplet localizes on a matter curve, the relative normalization of the kinetic terms between different components of the GUT multiplet will in general be different. Moreover, we give a qualitative explanation for why the mass relations become increasingly distorted for the lighter generations.

Recall that in the minimal SU(5) GUT, the net hypercharge flux vanishes on curves supporting complete GUT multiplets. Indeed, the converse of this condition for the Higgs curves provides a qualitative explanation for why these fields do not fill out full GUT multiplets. Although the *average* hypercharge flux vanishes on chiral matter curves, the field strength will in general *not vanish pointwise*. Because the individual components of a GUT multiplet have different hypercharge, the corresponding wave functions will couple differently to this background flux leading to distinct zero mode wave functions. The fact that the zero mode wave functions are not the same, and may in particular have different magnitudes, can be interpreted as Aharanov-Bohm interferences in a varying B-field background.

In a minimal SU(5) GUT, all of the interaction terms originate from evaluating the wave functions at points of triple intersection and now there is no reason why the magnitude of different matter fields within a GUT multiplet are the same. This leads to different Yukawa couplings and thus to different mass relations. In particular, assuming for simplicity no mixing between generations, we have modified mass relations of the form:

$$m_q = m_l \left| \frac{\psi_q(0)}{\psi_l(0)} \right| . \tag{14.16}$$

It would be interesting to examine whether modified GUT mass relations for the lighter generations of the general type proposed in [74] admit a geometric interpretation.

We now estimate the expected distortion in the usual GUT mass relations due to the Aharanov-Bohm effect with a varying B-field. To this end, let F_Σ denote the internal U(1) hypercharge field strength on the matter curve Σ. The overall scaling dependence of the mass relation violation δ can be determined by rescaling the overall volume of Σ by ε. Because the reduction of the instanton to Σ scales as $|F_\Sigma|^2/\varepsilon$, it follows that F_Σ rescales by a factor of $\sqrt{\varepsilon}$. This reduction is explained in further detail in [77]. It now follows that the violation of the mass relation will be proportional to:

$$\delta \sim \sqrt{\varepsilon}. \tag{14.17}$$

Note that as the volume of Σ tends to zero, the amount of violation in the mass relation also vanishes. Equation (4.25) implies that the masses of fields localized on Σ scale as:

$$M \sim 1/Vol(\Sigma) \sim 1/\varepsilon, \tag{14.18}$$

because in a canonical normalization of all fields, each wave function contributes a factor of $\psi(0)/\sqrt{M_*^2 Vol(\Sigma)}$ to the Yukawa couplings. Hence, the violation of the mass relation obeys the scaling law:

$$\delta \sim 1/\sqrt{M}. \tag{14.19}$$

While a mass relation will still hold for each generation, the particular numerical coefficient relating the masses will depend on the generation in question.

To conclude this section, we note that a common theme running throughout much of this paper is the central role of the internal hypercharge flux. Indeed, an intra-generational distortion in the usual GUT mass relations *requires* the presence of an internal hypercharge flux. In a sense, we can view the violation of the GUT mass relation as the first experimental evidence for the existence of extra dimensions!

14.4 Generating semi-realistic hierarchies and mixing angles

In this subsection we speculate on one possible way to achieve semi-realistic mass hierarchies and mixing angles in the context of our compactification. To frame the discussion to follow, we first review the field theory Froggatt-Nielsen Mechanism for generating a hierarchical structure in both the masses and mixing angles of the quark sector. As observed in [78], this naturally occurs when the up and down Yukawa couplings assume the form:

$$\lambda_{ij}^u = g_{ij}^u \varepsilon^{a_i+b_j},\ \lambda_{ij}^d = g_{ij}^d \varepsilon^{a_i+c_j}, \tag{14.20}$$

where the g's are order one 3×3 matrices and ε is a small parameter which is related to the Cabbibo angle $\theta_c \sim 0.2$. With this ansatz, the quark sector exhibits hierarchical masses and mixing angles determined by appropriate powers of ε [78].

From a field theory perspective, this type of power law suppression naturally occurs in theories with additional global U(1) symmetries. For example, if the superfields Q^i, U^i and D^i have charges a_i, b_i and c_i under a global U(1) symmetry, then the corresponding fields interact by coupling to an appropriate power of a gauge singlet charged under this global U(1). For example, letting ϕ denote a gauge singlet superfield with charge +1 under this global symmetry, the lowest order coupling in the superpotential is given by:

$$W \supset g_{ij}^u \left(\frac{\phi}{M_{\rm pl}}\right)^{-a_i-b_j} Q^i U^j H_u + g_{ij}^d \left(\frac{\phi}{M_{\rm pl}}\right)^{-a_i-c_j} Q^i D^j H_d \tag{14.21}$$

where for the purposes of this discussion we assume that H_u and H_d are neutral under the global U(1) symmetry. When ϕ develops a vev less than $M_{\rm pl}$, we obtain the expected hierarchy in the Yukawa couplings of equation (14.20).

We now speculate as to how such a hierarchy could potentially occur in compactifications of F-theory. Given a sufficiently generic configuration of matter curves which form triple intersections, in a holomorphic basis of wave functions the resulting holomorphic Yukawa couplings introduced in section 4 will be given by order one complex numbers. To extract the values of the physical up and down type Yukawa couplings, all of these fields must be rescaled to a canonical normalization of all kinetic terms. In the large volume limit, this simply rescales each wave function by an appropriate power of the overall volume factor so that the up and down type Yukawa couplings are:

$$\lambda_{ij}^u = g_{ij}^u Z_i^{(10)} Z_j^{(10)} Z_{H_u}, \, \lambda_{ij}^d = g_{ij}^d Z_i^{(10)} Z_j^{(\overline{5})} Z_{H_d} \tag{14.22}$$

where we have introduced the notation $Z = \left(M_*^2 Vol(\Sigma)\right)^{-1/2}$. In the above, the superscript on each Z denotes the representation and as usual, the indices i and j label the generations. In the extreme case where the volumes of the matter curves are hierarchical, this would provide a crude analogue of the Froggatt-Nielsen mechanism. It is not clear to us, however, that such a hierarchy is always available for self-intersecting curves. Indeed, it is likely that the Z's differ by order one factors. While this is typically enough to sufficiently distort the usual GUT mass relations, it may prove insufficient to produce the large hierarchy in mass scales between the top quark and the charm quark, for example.

Implicit in the above discussion is the assumption that the Z's of equation (14.22) only depend on the classical volumes of the matter curves. Indeed, as explained in section 4, the overall normalization of each wave function will receive quantum corrections away from the large volume limit. While we do not have a systematic method for computing these corrections, experience in perturbative string theory strongly suggests that these corrections are exponentially suppressed as functions of the Kähler moduli. Moreover, these corrections may induce small off-diagonal terms in the Kähler metric for the fields of the required type to generate a hierarchical structure in the physical Yukawa couplings.

In a similar vein, it is also tempting to speculate that non-perturbative contributions to the superpotential from Euclidean 3-branes wrapping divisors in the Calabi-Yau fourfold base could also contribute to a viable model of flavor physics. Indeed, because such corrections will typically violate global U(1) symmetries present in the low energy effective theory, the corresponding exponential factor can in principle have a form compatible with the Froggatt-Nielsen mechanism. While these remarks are admittedly speculative, it would be interesting to see whether there exist calculable examples of the desired type.

14.5 Textures from discrete symmetries and large diffeomorphisms

Discrete symmetries provide another possible way to achieve semi-realistic Yukawa couplings and interaction terms because such models can mimic the primary features of the Froggatt-Nielsen mechanism, but with the global continuous symmetry replaced by a discrete symmetry. In this approach, it is common to search for finite groups which admit two- and three-dimensional irreducible representations. For example, the two lightest generations could transform in a two-dimensional representation while the heaviest generation could transform as a singlet. As one application, these symmetries are typically enough to alleviate potential problems with FCNCs in gravity mediation scenarios.[19] A list of candidate discrete flavor groups with order at most thirty one which are of phenomenological interest has been tabulated in [79]. Some common choices in the model building literature are the symmetric group on three or four letters denoted by S_3 and S_4 as well as A_4, the alternating subgroup of S_4. See [80, 81] for a recent review of some possibilities along these lines. In the present context, the group of large diffeomorphisms of a del Pezzo surface provide a potentially attractive starting point for a theory of flavor based on discrete symmetries. We note that some version of this gauged symmetry will survive even away from the large volume regime. It is therefore possible that such symmetries could undergird a theory of flavor.

The group of large diffeomorphisms for the del Pezzo surfaces has a natural action on the matter curves of the del Pezzo which automatically lifts to a group action on the matter fields of the MSSM. For example, the del Pezzo 3 surface corresponds to the exceptional group $E_3 = \mathrm{SU}(3) \times \mathrm{SU}(2)$ which has Weyl group $S_3 \times S_2$. The S_3 factor could potentially play the role of the desired flavor group.

One potential caveat to the above proposal is that the action of the Weyl group on the matter curves corresponds to an integral representation. In other words, the corresponding characters take values in the integers. This follows from the fact that the Weyl group naturally permutes the exceptional curves of the del Pezzo surface. In particular, because the entries in the character tables for the phenomenologically most interesting representations of A_4 and S_4 are given by various powers of a third root of unity, this direct application of discrete symmetries may be too trivial.

We note that no similar obstruction is present in the case of the discrete group S_3. Indeed, consider as a toy model the case where the three generations have localized on the exceptional curves E_1, E_2 and E_3 of the del Pezzo 3 surface. In this case, the S_3 Weyl group permutes the exceptional curves. The three dimensional representation spanned by the three curves also determines how S_3 acts on the three generations. This three dimensional representation decomposes to the sum of a two dimensional representation and singlet which are respectively spanned by:

$$\langle E_1, E_2, E_3 \rangle \simeq \langle E_1 - E_2, E_2 - E_3 \rangle_{\mathrm{doublet}} \oplus \langle E_1 + E_2 + E_3 \rangle_{\mathrm{singlet}} . \tag{14.23}$$

This suggests that the wave function for the heavy generation transforms as the singlet, while the two light generations transform as the doublet. It would be interesting to develop such a theory of flavor in more detail.

[19] We thank K.S. Babu for emphasizing this point to us.

15. Suppression factors from singlet wave functions

So far we have only considered contributions to the superpotential from matter fields which all transform as non-trivial representations of G_{std}. A fully realistic model will most likely contain contributions to the effective superpotential from chiral superfields which transform as gauge singlets under G_{std}. For example, the μ term could originate from a cubic interaction term between the Higgs fields and a gauge singlet. The vev of this singlet would then set the size of μ. As another example, we note that because neutrino oscillations are now well-established, the superpotential must contain terms of the form $LN_R H_u$ where N_R denotes the right-handed neutrino superfields which transform as gauge singlets.

Generating appropriately small neutrino masses as well as a value for the μ term near the scale of electroweak symmetry breaking has historically been a challenge in string-based models. Some discussion on neutrino masses in string theory may be found for example in [82]. In type II D-brane constructions, contributions to the superpotential from wrapped Euclidean branes can produce an appropriately large Majorana mass term for right-handed neutrinos [54, 83]. Similar effects may also generate exponentially suppressed μ terms [54]. More recently, it has also been shown that D-brane instantons can also potentially generate suppressed Dirac neutrino masses [84]. In this section, we show that the Yukawa couplings which involve a singlet of G_S can in suitable circumstances be exponentially suppressed relative to the Yukawa couplings which only involve fields charged under G_S.

The rest of this section is organized as follows. In subsection 15.1, we study the behavior of gauge singlet wave functions which contribute to the low energy superpotential. After performing this analysis, in subsection 15.2, we estimate the overall normalization of the Yukawa couplings for such gauge singlet wave functions. For interaction terms involving three singlets, there is a natural volume suppression effect. For gauge singlets which are attracted to the GUT model seven-brane, the wave function behaves as if it had localized on a matter curve inside of S. For gauge singlets which are repelled away from the GUT model seven-brane, we find that the Yukawa couplings are naturally suppressed. In the remaining subsections we show that these effects can naturally generate both hierarchically small μ terms and neutrino masses. In both cases, we find that order one parameters in the high energy theory naturally can yield values which are in rough agreement with observation.

15.1 Wave function attraction and repulsion

To setup notation, we consider three seven-branes which wrap surfaces S, S', and S'' inside the compactification threefold B_3 and which carry respective gauge groups G_S, $G_{S'}$, and $G_{S''}$. By assumption, S, S', and S'' intersect transversely along smooth curves

$$\Sigma_X = S \cap S', \qquad \Sigma_Y = S \cap S'', \qquad \Sigma_\perp = S' \cap S'', \tag{15.1}$$

which give rise to corresponding chiral superfields X, Y, and Φ in four dimensions. Each superfield transforms as a bifundamental[20] under the respective products $G_S \times G_{S'}$, $G_S \times G_{S''}$,

[20] See §4.2 of [15] for a description of the generalized notion of "bifundamental" relevant for intersecting seven-branes in F-theory.

and $G_{S'} \times G_{S''}$. Finally, if the curves Σ_X, Σ_Y, and $\Sigma_\perp$ themselves intersect transversely at a single point, the low-energy effective superpotential contains a cubic coupling of the form

$$W_\perp = \lambda \, \Phi X Y \,, \tag{15.2}$$

invariant under $G_S \times G_{S'} \times G_{S''}$.

By assumption, the kinetic terms for X, Y, and Φ have the canonical normalization in four dimensions, so the dimensionless coupling λ in $W_\perp$ depends upon the L^2-norms of the associated zero-mode wavefunctions on the curves in (15.1). Since both Σ_X and Σ_Y are compact curves inside S, the norms of wavefunctions for X and Y merely scale with the volumes of the curves in S. However, unlike Σ_X and Σ_Y, the curve $\Sigma_\perp$ is not embedded in S but rather intersects S transversely at a point in B_3. From the perspective of the four-dimensional effective theory, this distinction in geometry is reflected by the fact that Φ transforms as a singlet under G_S, whereas X and Y form a vector-like pair. We are interested in the limit that S contracts inside B_3, or equivalently, in the limit that the volume of $\Sigma_\perp$ goes to infinity. In the limit that $\Sigma_\perp$ becomes non-compact, we clearly need to be careful in our estimate for the norm of the wavefunction ψ associated to the singlet Φ.

We are ultimately interested in the behavior of ψ near the point where $\Sigma_\perp$ intersects S, so let us introduce local holomorphic and anti-holomorphic coordinates $(z, \overline{z})$ on $\Sigma_\perp$ such that $z = 0$ is the location of the intersection with S. As we reviewed in section 3, ψ generally transforms on $\Sigma_\perp$ as a holomorphic section of the bundle $K_{\Sigma_\perp}^{1/2} \otimes L$,

$$\psi \in H^0_{\overline{\partial}}\big(\Sigma_\perp, K_{\Sigma_\perp}^{1/2} \otimes L\big) \,, \qquad\qquad L = L'\big|_{\Sigma_\perp} \otimes L''\big|_{\Sigma_\perp} \,, \tag{15.3}$$

where L' and L'' are line bundles on S' and S''. Because ψ is holomorphic, ψ satisfies

$$\overline{\partial}^\dagger \overline{\partial} \psi = 0 \,, \tag{15.4}$$

where $\overline{\partial}$ is the Dolbeault operator acting on $K_{\Sigma_\perp}^{1/2} \otimes L$, and $\overline{\partial}^\dagger$ is the adjoint operator defined with respect to the induced metric on $\Sigma_\perp$ and the hermitian metric on L inherited from L' and L''.

Besides the Dolbeault operator $\overline{\partial}$, the bundle $K_{\Sigma_\perp}^{1/2} \otimes L$ also carries a unitary connection which defines a covariant derivative ∇ and an associated Laplacian $\triangle = \nabla^\dagger \nabla$. By a standard Hodge identity reviewed in appendix E of [15], the Laplacian $\triangle$ is related to the operator $\overline{\partial}^\dagger \overline{\partial}$ via

$$\triangle = 2 \, \overline{\partial}^\dagger \overline{\partial} - \frac{1}{2} \mathcal{R} + \mathcal{F} \,. \tag{15.5}$$

Here $\mathcal{R}$ is the scalar curvature of the metric on $\Sigma_\perp$, and $\mathcal{F}$ is the scalar curvature of the unitary connection on L.

The positive constants in (15.5) will not be important for the following analysis, but the signs will be essential. First, the relative sign between $\mathcal{R}$ and $\mathcal{F}$ in (15.5) arises because $\mathcal{R}$ is the scalar curvature of the induced metric on $\Sigma_\perp$ and hence is the curvature of a connection on the holomorphic tangent bundle $T\Sigma_\perp \cong K_{\Sigma_\perp}^{-1}$, as opposed to a connection on the spin[21]

[21] The factor '1/2' multiplying $\mathcal{R}$ in (15.5) arises from the square-root in the spin bundle $K_{\Sigma_\perp}^{1/2}$.

bundle $K_{\Sigma_\perp}^{1/2}$. To fix the overall sign multiplying $\mathcal{R}$, we note that the Laplacian $\triangle$ is a positive-definite hermitian operator. On the other hand, because ψ is holomorphic,

$$\triangle\psi = \left(-\frac{1}{2}\mathcal{R} + \mathcal{F}\right)\psi\,. \tag{15.6}$$

According to (15.6), if $\mathcal{F} = 0$ and $\mathcal{R} > 0$ is strictly positive, then ψ must vanish. Such a vanishing is consistent with the fact that $K_{\Sigma_\perp}^{1/2} = \mathcal{O}(-1)$ admits no holomorphic sections on $\Sigma_\perp = \mathbb{P}^1$, and this observation fixes the sign of $\mathcal{R}$ in the Hodge identity (15.5).

In a local unitary frame, the Laplacian $\triangle$ takes the standard Euclidean form $\triangle = -4\,\partial^2/\partial z\,\partial\overline{z}$, and (15.6) reduces to the wave equation

$$4\,\frac{\partial^2\psi}{\partial z\partial\overline{z}} + \left(\mathcal{F} - \frac{1}{2}\mathcal{R}\right)\psi \,=\, 0\,. \tag{15.7}$$

Thus if ψ is normalized so that $\psi(0) = 1$, then ψ behaves near $z = 0$ as

$$\begin{aligned}\psi(z,\overline{z}) \,&=\, \exp\!\left(-\frac{1}{4}\,m_0^2\,|z|^2\right) \,+\, \cdots\,,\\ m_0^2 \,&=\, \left[\mathcal{F} - \frac{1}{2}\mathcal{R}\right]_{z=0}\,,\end{aligned} \tag{15.8}$$

where the '$\cdots$' indicate terms in ψ that vanish at $z = 0$, and the curvatures which define m_0^2 are evaluated at that point. In general, m_0^2 can be either[22] negative or positive, and the sign of m_0^2 determines whether ψ exponentially grows or decays away from the origin.

At first glance, one might be perplexed as to how such exponential behavior in ψ can arise, since nothing so far really distinguishes the point $z = 0$. In fact, given that ψ is written in a unitary frame, the behavior in (15.8) merely reflects the behavior of the metric on $K_{\Sigma_\perp}^{1/2}\otimes L$.

As a very concrete example, let us take $\Sigma_\perp$ to be $\mathbb{P}^1$, with a metric which we parameterize in Liouville form as

$$ds^2 \,=\, \mathrm{e}^{2\,\phi(z,\overline{z})}\,dz\,d\overline{z}\,. \tag{15.9}$$

For instance, if the metric on $\mathbb{P}^1$ is round with constant curvature Λ^2, then

$$\phi(z,\overline{z}) \,=\, -\ln\!\left(1 + \frac{1}{4}\,\Lambda^2|z|^2\right). \tag{15.10}$$

The role of the particular line bundle $K_{\Sigma_\perp}^{1/2}\otimes L$ is inessential, so for simplicity we just take ψ to transform in the holomorphic tangent bundle $T\mathbb{P}^1$. As is well-known, holomorphic tangent vectors on $\mathbb{P}^1$ take the global form

$$u(z)\,\frac{\partial}{\partial z}\,, \qquad\qquad u(z) \,=\, a_0 + a_1 z + a_2 z^2\,, \tag{15.11}$$

[22] Holomorphy of ψ implies that the total curvature satisfies $\int_{\Sigma_\perp}\star(\mathcal{F} - \frac{1}{2}\mathcal{R}) \geq 0$, but the sign of $\mathcal{F} - \frac{1}{2}\mathcal{R}$ may vary from point to point on $\Sigma_\perp$.

where (a_0, a_1, a_2) are complex parameters. However, if $\phi(z, \overline{z})$ in (15.9) varies non-trivially over $\mathbb{P}^1$, the holomorphic vector $\partial/\partial z$ does not have constant length. To describe ψ in a unitary frame, we instead introduce a new basis vector $\hat{e}$ for $T\mathbb{P}^1$,

$$\hat{e} \,=\, \frac{1}{2}\,\mathrm{e}^{-\phi(z,\overline{z})}\,\frac{\partial}{\partial z}\,. \tag{15.12}$$

Though $\hat{e}$ is not holomorphic, $\hat{e}$ does have constant, unit length in the metric (15.9). In the frame described by $\hat{e}$, a holomorphic tangent vector ψ therefore takes the form

$$\psi \,=\, \mathrm{e}^{\phi(z,\overline{z})}\,u(z)\,\hat{e}. \tag{15.13}$$

Because the scalar curvature of the metric in (15.9) is given in terms of ϕ as

$$\mathcal{R} \,=\, -4\,\mathrm{e}^{-2\,\phi}\,\frac{\partial^2\phi}{\partial z\,\partial\overline{z}}\,, \tag{15.14}$$

the behavior near $z = 0$ of ψ in (15.13) is controlled by the local curvature.[23]

To make use of (15.8), we must still estimate m_0^2 at the point where $\Sigma_\perp$ intersects the surface S. Since m_0^2 receives contributions from both $\mathcal{R}$ and $\mathcal{F}$, we consider each contribution in turn.

To estimate $\mathcal{R}$, we recall that S is a del Pezzo surface shrinking to zero size inside the elliptic Calabi-Yau fourfold $\mathcal{X}$. As a result, the scalar curvature on S is large and positive, of order M_{GUT}^2. On the other hand, since $\mathcal{X}$ is Calabi-Yau, the total scalar curvature on $\mathcal{X}$ vanishes. Because the elliptic fiber of $\mathcal{X}$ is generically non-degenerate, with negligible curvature, the large positive curvature of S near its point of intersection with $\Sigma_\perp$ must be locally cancelled by a corresponding negative curvature on $\Sigma_\perp$ itself. The scalar curvature $\mathcal{R}$ on $\Sigma_\perp$ near $z = 0$ is thus negative and of order

$$\mathcal{R} \,\sim\, -M_{\mathrm{GUT}}^2\,. \tag{15.15}$$

We note that if $\Sigma_\perp$ has genus zero or one, then $\mathcal{R}$ must become positive elsewhere on $\Sigma_\perp$ as dictated by the Euler characteristic.

We apply a similar argument to estimate the curvature $\mathcal{F}$ on L near $z = 0$. By definition, the line bundle L is a tensor product $L'\big|_{\Sigma_\perp} \otimes L''\big|_{\Sigma_\perp}$ of line bundles L' and L'' on respective surfaces S' and S'', and both L' and L'' carry anti-self-dual connections. The following observations are symmetric between L' and L'', but for concreteness let us focus on the bundle L' over S'.

The surface S' contains two curves $\Sigma_X = S \cap S'$ and $\Sigma_\perp = S'' \cap S'$ which intersect transversely at the point $z = 0$ on $\Sigma_\perp$. Since S is shrinking inside $\mathcal{X}$, the curve Σ_X is similarly shrinking inside the surface S'. In this situation, an anti-self-dual connection on L' over S' must restrict to a solution of the two-dimensional Yang-Mills equations on the shrinking curve Σ_X. Hence the curvature of L' on Σ_X must be constant and uniform, of order $d\mathrm{Vol}(\Sigma_X)^{-1} \sim d\,M_{\mathrm{GUT}}^2$, where d is the degree of L' on Σ_X.

[23]Because of the conventions adopted, $\mathcal{R}$ in (15.14) plays the role of $\mathcal{F}$ in (15.8).

Without loss, we assume that the metric on S' at the intersection of Σ_X and $\Sigma_\perp$ takes the diagonal form $ds^2 = dzd\overline{z} + dwd\overline{w}$, where w is a local holomorphic coordinate on Σ_X and z is a local holomorphic coordinate on $\Sigma_\perp$. Because the curvature of the connection on L' is anti-self-dual, the curvature at $z = 0$ along $\Sigma_\perp$ must be opposite to the curvature along Σ_X. Hence the curvature of L' on $\Sigma_\perp$ is of order $-d\, M^2_{\mathrm{GUT}}$.

Including a similar contribution from L'', we find

$$\mathcal{F} \sim -\left[\frac{\deg(L'|_{\Sigma_X})}{\mathrm{Vol}(\Sigma_X)} + \frac{\deg(L''|_{\Sigma_Y})}{\mathrm{Vol}(\Sigma_Y)}\right] \sim \pm M^2_{\mathrm{GUT}}\,. \tag{15.16}$$

Both $\mathcal{R}$ and $\mathcal{F}$ are of roughly the same magnitude, but whereas the sign of $\mathcal{R}$ is fixed, the sign of $\mathcal{F}$ generally depends upon the degrees of L' and L'' as well as the relative volumes of the matter curves Σ_X and Σ_Y in S. We see no particular reason why the contributions to $\mathcal{F}$ from Σ_X and Σ_Y should be correlated in either sign or absolute value. So depending upon the choices for L' and L'', the parameter $m_0^2 = \mathcal{F} - \frac{1}{2}\mathcal{R}$ can be either positive or negative, of order M^2_{GUT}.

We are left to estimate the norm of the singlet wavefunction ψ. Now, the great virtue of writing ψ in a unitary frame is that the L^2-norm of ψ is given directly by

$$||\psi||^2 = M_*^2 \int_{\Sigma_\perp} \omega\, |\psi|^2\,, \qquad \omega = \frac{i}{2}\,\mathrm{e}^{2\phi(z,\overline{z})}\, dz\wedge d\overline{z}\,, \qquad |\psi|^2 \equiv \overline{\psi}\psi\,. \tag{15.17}$$

Here ω is the Kähler form for the induced metric on $\Sigma_\perp$, which for concreteness we parameterize in the Liouville form (15.9). According to (15.8) and (15.14), the integrand of (15.17) then behaves to leading order near $z = 0$ as

$$\begin{aligned}\mathrm{e}^{2\phi(z,\overline{z})}\,|\psi|^2 &\approx \exp\left[-\frac{1}{2}\,(m_0 + \mathcal{R})\,|z|^2\right],\\ &= \exp\left[-\frac{1}{2}\left(\mathcal{F} + \frac{1}{2}\mathcal{R}\right)|z|^2\right].\end{aligned} \tag{15.18}$$

If the combination $\mathcal{F} + \frac{1}{2}\mathcal{R}$ is positive at $z = 0$, the integral over $\Sigma_\perp$ in (15.17) has rapid Gaussian decay at the scale M_{GUT}, so immediately

$$||\psi||^2 \sim \frac{M_*^2}{M^2_{\mathrm{GUT}}}\,, \qquad\qquad \left[\mathcal{F} + \frac{1}{2}\mathcal{R}\right]_{z=0} > 0\,. \tag{15.19}$$

In this case the normal wave function is *attracted* to our brane.

Conversely, if $\mathcal{F} + \frac{1}{2}\mathcal{R}$ is negative at $z = 0$, the expression in (15.18) rapidly blows up away from the origin. In this case the normal wave function is *repelled* from our brane. To make sense of $||\psi||^2$, we impose a cutoff in the integral over $\Sigma_\perp$ at a scale $|z| \sim R_\perp$. As we discuss briefly below, we expect the Gaussian approximation in (15.18) to be valid up to the cutoff, so we estimate $||\psi||^2$ as

$$\langle\psi|\psi\rangle = ||\psi||^2 \sim \frac{M_*^2}{M^2_{\mathrm{GUT}}}\,\exp\bigl(c\,M^2_{\mathrm{GUT}}R^2_\perp\bigr)\,, \qquad\qquad \left[\mathcal{F} + \frac{1}{2}\mathcal{R}\right]_{z=0} < 0\,. \tag{15.20}$$

In this estimate, $c > 0$ is an order one constant which our analysis does not fix, though the expression in (15.20) depends sensitively upon its value. Similarly, the estimate depends upon our choice of $R_\perp$, which roughly encodes the behavior of the metric on B_3 away from S. We recall that $R_\perp$ is parameterized as

$$R_\perp = M_{\mathrm{GUT}}^{-1}\,\varepsilon^{-\gamma}\,, \qquad\qquad \varepsilon = \frac{M_{\mathrm{GUT}}}{\alpha_{\mathrm{GUT}} M_{\mathrm{pl}}}\,, \tag{15.21}$$

where γ typically lies in the range $1/3 < \gamma < 1$.

In making the estimate (15.20) for $||\psi||^2$, we assume that the curvature of the Calabi-Yau metric on $\mathcal{X}$ (and similarly the connection on L) is slowly varying and of order M_{GUT}^2 in a region of size $R_\perp$ away from S. This behavior of the Calabi-Yau metric on $\mathcal{X}$ is suggested by similar behavior of the local Calabi-Yau metric on the cotangent bundle $T^*\mathbb{CP}^1$, as exhibited for instance in §3 of [85]. In the case of $T^*\mathbb{CP}^1$, the scalar curvature $\mathcal{R}$ along the cotangent fiber experiences only a slow, power-law decay away from $\mathbb{CP}^1$, and we roughly expect the same behavior normal to S in $\mathcal{X}$. However, a more precise estimate of $||\psi||^2$ clearly demands a more detailed analysis of the local Calabi-Yau metric on $\mathcal{X}$.

15.2 Estimating Yukawa couplings

Having estimated the local behavior of gauge singlet wave functions near the del Pezzo surface, we now determine the corresponding values of the Yukawa couplings in the low energy theory. With notation as above, to estimate the size of the Yukawa coupling in equation (15.2), we introduce the wave function x (resp. y) for the chiral superfield X (resp. Y) which localizes on the matter curve Σ_X (resp. Σ_Y) in S. The superpotential term of equation (15.2) due to a triple overlap between Σ_X, Σ_X, $\Sigma_\perp$ at a point p is:

$$W_\perp = \lambda \Phi X Y \tag{15.22}$$

$$= \frac{x(p)}{\sqrt{M_*^2 Vol(\Sigma_X)}} \frac{y(p)}{\sqrt{M_*^2 Vol(\Sigma_Y)}} \frac{\psi(p)}{\sqrt{\langle\psi|\psi\rangle}} \Phi X Y \tag{15.23}$$

where in the above, we have adopted the physical normalization of Yukawa couplings detailed in section 4. The value of the Yukawa coupling strongly depends on whether the del Pezzo surface attracts or repels the gauge singlet wave function from the point of intersection. By contrast, we note that because X and Y localize on matter curves inside of S, the values of $x(p)$ and $y(p)$ are order one numbers. Making the rough approximation $M_*^2 Vol(\Sigma) \sim \alpha_{\mathrm{GUT}}^{-1/2}$, the resulting Yukawa coupling is:

$$\lambda = \alpha_{\mathrm{GUT}}^{1/2} \frac{\psi(p)}{\sqrt{\langle\psi|\psi\rangle}}. \tag{15.24}$$

We now estimate the value of the Yukawa coupling depending on whether the GUT model seven-brane attracts or repels the gauge singlet wave function. To this end, we shall frequently refer back to the estimates of the various length scales obtained in section 4. In the repulsive case, equation (15.20) now implies:

$$\lambda_{\mathrm{repel}} \sim \alpha_{\mathrm{GUT}}^{1/2} \times \left(\alpha_{\mathrm{GUT}}^{1/4} \frac{R_S}{R_\perp} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right)\right) \tag{15.25}$$

$$= \alpha_{\mathrm{GUT}}^{3/4} \times \varepsilon^{\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right) \tag{15.26}$$

where the second equality follows from equation (4.17) and as in the previous subsection, c is a positive order one number.

By contrast, in the undamped case described by equation (15.19), the associated Yukawa coupling is:

$$\lambda_{\mathrm{attract}} \sim \alpha_{\mathrm{GUT}}^{1/2} \frac{M_{\mathrm{GUT}}}{M_*} \sim \alpha_{\mathrm{GUT}}^{3/4}. \tag{15.27}$$

Physically, the value of $\lambda_{\mathrm{attract}}$ agrees with the intuition that in the attractive case, all details of the compactification decouple because the gauge singlet behaves as though it localizes on a curve in S. In general, we see that:

$$|\lambda_{\mathrm{attract}}| \gg |\lambda_{\mathrm{repel}}|. \tag{15.28}$$

In addition to interaction terms between matter fields inside of S and a single gauge singlet, it is also possible for three gauge singlet wave functions to interact outside of S. When one such gauge singlet develops a non-zero vev, the resulting interaction term will determine the mass of the remaining gauge singlets. Letting ψ_i denote gauge singlet wave functions for $i = 1, 2, 3$, the value of the physical Yukawa coupling from wave function overlap at a point b outside of S is now given by:

$$\lambda_{\mathrm{singlet}} \sim \frac{\psi_1(b)}{\sqrt{M_*^2 Vol(\Sigma_1)}} \frac{\psi_2(b)}{\sqrt{M_*^2 Vol(\Sigma_2)}} \frac{\psi_3(b)}{\sqrt{M_*^2 Vol(\Sigma_3)}} \sim \frac{1}{(M_* R_\perp)^3} \tag{15.29}$$

$$\sim \alpha_{\mathrm{GUT}}^{3/4} \left(\frac{R_S}{R_\perp}\right)^3 = \alpha_{\mathrm{GUT}}^{3/4} \times \varepsilon^{3\gamma}. \tag{15.30}$$

We note that in comparison to Yukawa couplings on S which are on the order of $\alpha_{\mathrm{GUT}}^{3/4}$, this naturally yields an overall suppression factor by a non-trivial power of ε.

15.3 μ term

We now discuss a natural mechanism for obtaining small supersymmetric μ terms. For concreteness, suppose that the bulk gauge group $G_S = SU(5)$ and that the H_u and H_d fields localize on distinct matter curves where the singularity type enhances to SU(6). In the case where these curves do not intersect, the μ term is automatically zero. In the case where they do intersect, the matter fields will interact with a gauge singlet which localizes on a curve normal to S. Letting Φ denote the chiral superfield for this gauge singlet, the superpotential now contains the interaction term:

$$W_\mu \supset \lambda \Phi H_u H_d \sim \alpha_{\mathrm{GUT}}^{1/2} \frac{\psi(p)}{\sqrt{\langle \psi | \psi \rangle}} \Phi H_u H_d \tag{15.31}$$

with notation as in equation (15.24). When Φ develops a vev, the superpotential will contain a μ term for the Higgs up and Higgs down fields. The value of this vev is controlled by the dynamics orthogonal to S and therefore scales as:

$$\langle \Phi \rangle \sim \frac{1}{R_\perp} \sim M_{\mathrm{GUT}} \times \varepsilon^{\gamma}. \tag{15.32}$$

Returning to equations (15.25) and (15.27), it thus follows that in the attractive case, the resulting value of μ is far above the electroweak scale, and would lift the Higgs doublets from the low energy spectrum. On the other hand, in the exponentially damped case, the value of the μ term is:

$$\mu = \lambda_{\text{repel}} \langle \Phi \rangle \sim \alpha_{\text{GUT}}^{3/4} \times \varepsilon^{2\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right). \tag{15.33}$$

This leads to a large hierarchy between the μ term and the GUT scale. For example, with $\gamma = 1$ and $c = 1/7$ we find $\mu \sim 140\,\text{GeV}$. In section 19 we present some additional estimates of μ.

15.4 Neutrino masses

At a conceptual level, the μ term and Dirac mass terms for the neutrinos both originate from interactions between two fields on curves in S and a third field which localizes on a curve normal to S. Indeed, in the previous subsection we found that when the gauge singlet wave function is exponentially suppressed near S, the μ term is hierarchically suppressed below the GUT scale. We now estimate the values of the light neutrino masses of the MSSM depending on the profile of the right-handed neutrino wave function near the surface S. When the gauge singlet is attracted to S, a variant on the usual seesaw mechanism yields neutrino masses which are approximately correct. On the other hand, when the gauge singlet is repelled away from S, the value of the Dirac masses is already quite low, and the seesaw mechanism would yield unviable neutrino masses. In fact, the Dirac mass terms are already in a viable range so that in this case the neutrinos are purely of Dirac type.

For simplicity, we perform our estimates for a single neutrino species, because as explained in section 14, a detailed model of flavor is currently beyond our reach. In this case, the neutrino sector of the superpotential is:

$$W_\nu = \lambda_D L N_R H_u + \lambda_{\text{singlet}} \Theta N_R N_R \tag{15.34}$$

where N_R denotes the right-handed neutrino chiral superfield, and Θ is another gauge singlet. In certain cases, the second interaction term may not be present. In the following we analyze the interplay between the behavior of the right-handed neutrino wave functions near S and this second interaction term.

15.4.1 Majorana masses and a seesaw

We now consider the case where the second interaction term $\Theta N_R N_R$ does not vanish and show that a phenomenologically viable scenario requires that the right-handed neutrino wave function is attracted to S. When Θ develops a vev, it induces a Majorana mass term for the right-handed neutrinos. Using the value of λ_{singlet} given by equation (15.30), this yields the Majorana mass:

$$m_M \equiv \lambda_{\text{singlet}} \langle \Theta \rangle = \frac{\lambda_{\text{singlet}}}{R_\perp} = \alpha_{\text{GUT}}^{3/4} M_{\text{GUT}} \times \varepsilon^{4\gamma} \sim 3 \times 10^{12\pm1.5}\ \text{GeV}. \tag{15.35}$$

The value of the Dirac masses strongly depends on the profile of the gauge singlet wave function near S. By inspection of equations (15.25) and (15.27), the value of λ_{attract} will induce a Dirac mass term for neutrinos which is around the electroweak scale, while the value of λ_{repel} will induce a far smaller Dirac mass term. The mass matrix for the neutrinos is:

$$M_\nu = \begin{bmatrix} 0 & \frac{1}{2}m_D \\ \frac{1}{2}m_D & m_M \end{bmatrix} \sim \alpha_{\text{GUT}}^{3/4} \begin{bmatrix} 0 & \langle H_u \rangle \\ \langle H_u \rangle & M_{\text{GUT}} \times \varepsilon^{4\gamma} \end{bmatrix}. \tag{15.36}$$

Because the Majorana mass term is non-zero, it is much larger than the Dirac mass terms so that the smaller eigenvalue of M_ν is given by the usual seesaw mechanism:

$$m_{\text{light}} \sim \frac{m_D^2}{m_M}. \tag{15.37}$$

Due to the fact that the Majorana mass term is in the usual range expected for a seesaw mechanism, m_D must be on the order of the electroweak scale in order to yield a viable light neutrino mass. Restricting to this case, m_{light} is now given by:

$$m_{\text{light}} \sim \left(\alpha_{\text{GUT}}^{3/4} \times \varepsilon^{-4\gamma}\right) \times \frac{\langle H_u \rangle^2}{M_{\text{GUT}}} \sim 2 \times 10^{-1\pm1.5} \text{ eV}. \tag{15.38}$$

We note that in this case, we automatically find an enhancement over the naive seesaw value $\langle H_u \rangle^2 / M_{\text{GUT}}$! Indeed, in the GUT literature it is often necessary to lower the Majorana mass term below M_{GUT} to obtain more realistic neutrino masses.

15.4.2 Suppressed Dirac masses

Next consider the possibility that the interaction term between Θ and N_R in equation (15.34) does not exist so that the neutrinos are purely of Dirac type. In the previous subsection we found that a variant of the standard seesaw mechanism requires that the right-handed neutrino wave function is attracted towards S. Indeed, the Dirac mass terms for the undamped wave functions were automatically on the order of the electroweak scale. In the absence of a seesaw mechanism, this profile for the wave functions would yield an unacceptably large value for the neutrino masses. On the other hand, the wave functions which are repelled away from S will naturally generate much smaller Dirac neutrino mass terms.

Restricting to the repulsive case, the Dirac mass term is:

$$m_{\text{Dirac}} = \lambda_{\text{repel}} \langle H_u \rangle \sim \langle H_u \rangle \times \left[\alpha_{\text{GUT}}^{3/4} \times \varepsilon^{\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right)\right]. \tag{15.39}$$

The essential point of the above formula is that the Dirac mass can be quite light, and for an appropriate order one value of c, yields a phenomenologically viable mass for the light neutrinos. For example, setting $c = 5$ and $\gamma = 1/3$ yields $m_{\text{Dirac}} \sim 6 \times 10^{-3}$ eV. Before closing this subsection, we note that while large Majorana mass terms which violate lepton number are typically invoked as a primary cause of leptogenesis in early universe cosmology, there do exist viable alternative scenarios which only require Dirac neutrino masses. See [86] and references therein for a recent account of Dirac leptogenesis.

15.5 Relating μ and ν

In the previous subsection we presented a general formula which naturally generates an exponentially suppressed value for the masses of purely Dirac type neutrinos. Indeed, the exponential damping terms for both the μ term of equation (15.33) and the Dirac mass term of equation (15.39) are both sensitive to an order one parameter which we denote by c. We now present a relation between μ and m_{Dirac} in which the overall dependence on this exponential factor cancels out. This expression is model independent in the sense that it does not depend as strongly on the details of the exponential suppression factor.

The exponential suppression factors of the μ term and the purely Dirac mass term both originate from a gauge singlet wave function which is repelled away from the surface S so that:

$$m_{\text{Dirac}} = \lambda_{\text{repel}}(c) \langle H_u \rangle \tag{15.40}$$

$$\mu = \lambda_{\text{repel}}(c') \langle \Phi \rangle \tag{15.41}$$

where $\langle \Phi \rangle$ denotes the vev of a gauge singlet which localizes on a matter curve normal to S. In the above, we have allowed two potentially different suppression factors such that c and c' may differ by some small amount.

Making the simplifying assumption $c = c'$, all exponential effects cancel, and we obtain the rough estimate:

$$m_{\text{Dirac}} = \mu \frac{\langle H_u \rangle}{\langle \Phi \rangle} = \frac{\mu \varepsilon^{-\gamma}}{\langle H_u \rangle} \times \frac{\langle H_u \rangle^2}{M_{\text{GUT}}} \sim 5 \times 10^{-3 \pm 0.5} \text{ eV} \tag{15.42}$$

for $\mu \sim 100\,\text{GeV}$. Of course, for small mismatches between the parameters c and c', slightly higher (or lower) values are also in principle possible.

16. Supersymmetry breaking

Up to now, our analysis has assumed that the four-dimensional effective theory preserves $\mathcal{N} = 1$ supersymmetry. See [87, 88] for recent discussions of supersymmetry breaking in F-theory and [89] for an explicit realization of gauge mediated supersymmetry breaking in an intersecting D-brane model. In this section we briefly sketch how supersymmetry breaking can be communicated to the MSSM in a gauge mediation scenario. Further details will appear in [90]. A more general framework which interpolates between gauge mediation and gravity mediation is given in [52]. In that context, supersymmetry breaking takes place on a seven-brane distinct from a GUT model seven-brane. When these branes intersect, supersymmetry breaking is communicated via gauge mediation. As the separation between the seven-branes increases, this interpolates to a gravity mediation scenario. In the present case, most of our seven-branes form non-trivial topological intersections which cannot disappear. While we shall present some brief speculations on generating hierarchically small values for the scale of supersymmetry breaking, a complete analysis would entail a broader discussion which is beyond the scope of this paper.

To frame the discussion to follow, we now briefly sketch the basic features of gauge mediated supersymmetry breaking. See [91] for a review of gauge mediation. In general, most mediation mechanisms consist of three sectors. These are given by the sector of the theory which breaks supersymmetry, the sector of communication, and the MSSM itself. Although we do not specify how supersymmetry can be broken, we can still parameterize this breaking in terms of at least one chiral superfield X which develops a supersymmetry breaking vev:

$$\langle X \rangle = x + \theta^2 F. \tag{16.1}$$

To specify the messenger sector, we introduce vector-like pairs of GUT multiplets which will communicate supersymmetry breaking to the MSSM. As an explicit example, we take Y to transform in the fundamental of SU(5) and Y' in the anti-fundamental. These fields can then localize on matter curves inside of S. The messengers couple to X via an interaction term of the form:

$$W_{4d} \supset W_{\text{mess}} = \lambda X Y Y^{'}. \tag{16.2}$$

Once X develops a vev of the type given by equation (16.1), the messengers will get a mass:

$$M_{\text{mess}} = \lambda x. \tag{16.3}$$

Supersymmetry breaking then communicates to the MSSM because the messenger fields interact with the gauge bosons of the MSSM. In this setup, the soft masses for the gauginos are generated at one loop order while the soft scalar masses are generated at two loop order. One attractive feature of the gauge mediation scenario is that FCNCs are automatically suppressed.

Although precise numerical estimates are beyond the scope of the present paper, to simply get a sense of the mass scales involved, recall that in gauge mediation, the masses of the gauginos are:

$$m_i \sim \frac{\alpha_i(M_{\text{weak}})}{4\pi} \frac{F}{x}. \tag{16.4}$$

We note that this estimate does not require any knowledge of the overall normalization factors appearing in equation (16.2). The lightest gaugino in this case is the Bino which in viable models has a mass of $\sim 100\,\text{GeV}$. Plugging in the properly normalized value of the hypercharge coupling at the weak scale given by $\alpha_1(M_{\text{weak}}) \sim (5/3) \times (1/128) \sim 10^{-2}$, we see that the scale of supersymmetry breaking $\sqrt{F}$ and the messenger scale x are related via:

$$\sqrt{F} \sim 300 \text{ GeV}^{1/2} \sqrt{x}. \tag{16.5}$$

Depending on the origin of the X field in the F-theory GUT model, the resulting messenger mass scales can potentially be quite different. In the following subsections we discuss three natural candidates for X in the present class of compactifications. The field X can correspond to a bulk gauge boson on a transversely intersecting seven-brane, or a field which localizes on a matter curve orthogonal to S. In the latter case, there are two further refinements depending on whether the GUT model seven-brane attracts or repels the corresponding gauge singlet wave function.

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16.1 Bulk gauge boson breaking

When the matter fields Y and Y' localize on the same curve, these fields will automatically couple to the bulk gauge fields of a seven-brane which transversely intersects the GUT model seven-brane. In this case, we can interpret x as the supersymmetric vev of the bulk gauge field. The value of x depends on the volume of the matter curve containing the messenger fields as well as the remaining bulk worldvolume of the other seven-brane. Using the basic scaling relations obtained in section 4, we estimate $\langle X\rangle \sim 1/R_{\perp}$ so that the resulting messenger mass is:

$$M_{\text{mess}} = \alpha_{\text{GUT}} M_{\text{GUT}} \varepsilon^{2\gamma} \tag{16.6}$$

$$\sim 1 \times 10^{15\pm 0.5} \text{ GeV}. \tag{16.7}$$

16.2 Gauge singlet breaking

It is also possible that X could correspond to a gauge singlet which localizes on a matter curve which intersects S at a point. In this case, much of the analysis performed in section 15 carries over. For example when the gauge singlet wave function for X is attracted towards the seven-brane, it couples to the messenger fields with the same strength as a field inside of S. In this case, the messenger mass is on the order of:

$$M_{\text{mess}} = \frac{\lambda_{\text{attract}}}{R_{\perp}} = \alpha_{\text{GUT}}^{3/4} M_{\text{GUT}} \times \varepsilon^{\gamma} \sim 5 \times 10^{14\pm 0.5} \text{ GeV}. \tag{16.8}$$

On the other hand, the seven-brane can also repel the gauge singlet wave function. In this case, the messenger mass scale can be hierarchically much lighter than the GUT scale due to the exponential suppression factor present at the point of intersection with the seven-brane. In this case, the resulting messenger mass is given by a similar expression to that derived for the μ term in equation (15.33):

$$M_{\text{mess}} = \frac{\lambda_{\text{repel}}}{R_{\perp}} \sim M_{\text{GUT}} \times \alpha_{\text{GUT}}^{3/4} \varepsilon^{2\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right). \tag{16.9}$$

In this case, the messenger mass scale can potentially range over many candidate values. For example, we obtain a value of $\sim 10^{12}$ GeV when $c = 1$ and $\gamma = 1/3$, and a value of ~ 300 TeV when $c = 1/10$ and $\gamma = 1$.

16.3 Soft breaking boundary conditions

A well-known difficulty with the gauge mediation scenario is that it is typically difficult to simultaneously generate the correct values for the μ and $B\mu$ terms. In the present context, we note that the μ term is naturally light and on the order of the electroweak scale. Indeed, this setup decouples the issue of supersymmetry breaking from the μ problem. In fact, at the GUT scale, the $B\mu$ term is zero at high energies, and is instead radiatively generated. Phenomenological fits to this range of parameter space favor larger values for $\tan\beta = \langle H_u\rangle/\langle H_d\rangle$ [92].

We also expect that higher order terms in the superpotential of the form:

$$W_{\text{quart}} = \frac{c_{ijk}}{M_{KK}} X \Lambda_i \Lambda_j \Lambda_k \tag{16.10}$$

where Λ_i denotes a generic field of the MSSM cannot be generated by integrating out Kaluza-Klein modes. As explained in section 13, this is due to the fact that such terms will typically violate a global U(1) symmetry in the low energy theory. Indeed, matter fields in F-theory are always charged under additional U(1) factors of precisely this type. Letting σ_i denote the bosonic component of the chiral superfield Λ_i, this suggests that the values of the soft breaking A-terms in the effective potential:

$$V_{\text{eff}} = A_{ijk}\sigma_i\sigma_j\sigma_k \tag{16.11}$$

will automatically vanish at the scale set by x. Because both the $B\mu$ and A terms vanish, there is a common rephasing symmetry of the fields which naturally avoids additional CP violating phases.

16.4 Speculations on supersymmetry breaking

To conclude this section, we now briefly speculate on ways in which supersymmetry breaking can take place in the various scenarios outlined above. First consider the case where X is identified with a bulk gauge field on a seven-brane which intersects the GUT seven-brane. Returning to the equations of motion for fields on the transversely intersecting seven-brane S' derived in [15], the value of F is:

$$F^* = \overline{\partial}'\phi' + \delta_\Sigma \left\langle \langle Y^c, Y \rangle \right\rangle_{ad(P)} + \dots \tag{16.12}$$

where ϕ' denotes the holomorphic $(2,0)$ form for this brane and the ... denotes contributions to the F-term localized on other matter curves in the surface S'. When the righthand side of the above equation is non-zero, this will break supersymmetry. This can easily occur when the background value of the G-flux in the Calabi-Yau fourfold is incompatible with the complex structure on S'. Because this difference can be quite small in principle, we can obtain small values for F in this case.

Next consider scenarios where the X field corresponds to a gauge singlet localized on a matter curve intersecting S at a point. While we have primarily focussed on the behavior of this wave function in supersymmetric backgrounds, presumably a similar analysis will also carry through in a non-supersymmetric background. In this vein, it may be possible to extend the discussion of section 15 to this more general case. It would be interesting to see whether a suitable hierarchy in the scale of supersymmetry could be arranged in this way.

17. SU(5) model

Having presented a number of potential model building ingredients in the previous sections, we now proceed to some semi-realistic examples of models based on a del Pezzo 8 surface which incorporates at least some of these ideas. Our expectation is that significant refinements are possible in the actual examples we present. As explained in previous sections, the GUT group directly breaks to G_{std} via an internal hypercharge flux. Moreover, to avoid exotic matter representations, the available internal fluxes are in one to one correspondence with the roots of an exceptional Lie algebra. In this case, all of the matter content of the

MSSM must localize on curves in S. The fields in the 5 or $\overline{5}$ localize on curves where the bulk SU(5) singularity enhances to SU(6), while fields in the 10 and $\overline{10}$ localize on curves where SU(5) enhances to SO(10).

As explained in [15], the interaction terms $\overline{5}_H \times \overline{5}_M \times 10_M$ originate from points where the bulk singularity $G_S = \mathrm{SU}(5)$ undergoes a twofold enhancement in rank to an SO(12) singularity. Similarly, the interaction terms $5_M \times 10_M \times 10_M$ originate from a twofold enhancement in rank to an E_6 singularity. As in [15], we deduce the local behavior of the matter curves near such points by decomposing the adjoint representations of SO(12) and E_6 to the product $\mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1)$:

$$\mathrm{SO}(12) \supset \mathrm{SU}(5) \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \tag{17.1}$$

$$66 \to 1_{0,0} + 1_{0,0} + 24_{0,0} \tag{17.2}$$

$$+ 5_{2,2} + 5_{-2,2} + \overline{5}_{-2,2} + \overline{5}_{-2,-2} + 10_{4,0} + \overline{10}_{-4,0} \tag{17.3}$$

$$E_6 \supset \mathrm{SU}(5) \times \mathrm{U}(1)_a \times \mathrm{U}(1)_b \tag{17.4}$$

$$78 \to 1_{0,0} + 1_{0,0} + 1_{-5,-3} + 1_{5,3} + 24_{0,0} \tag{17.5}$$

$$+ 5_{-3,3} + \overline{5}_{3,-3} + 10_{-1,-3} + \overline{10}_{1,3} + 10_{4,0} + \overline{10}_{-4,0}. \tag{17.6}$$

Consider first the fields associated with the Cartan of SO(12). Labeling the local Cartan generators as t_1, t_2, we conclude that a six-dimensional field in the 5 localizes on the matter curve $(t_1 + t_2 = 0)$ and another field in the 5 localizes along $(t_1 - t_2 = 0)$, while a 10 localizes on the matter curve $(t_1 = 0)$. Similar considerations apply for E_6, from which we conclude that a six-dimensional field in the 5 localizes on the matter curve $(-t_a + t_b = 0)$, while distinct six-dimensional 10's localize on the matter curves $(t_a + 3t_b = 0)$ and $(t_a = 0)$. The gauge singlets of SU(5) localize on curves which only intersect S at a discrete set of points. To generate naturally suppressed μ terms and light Dirac masses for the neutrinos, we also consider local enhancements to SU(7).

For illustrative purposes, we first present an example which we shall refer to as "Model I" which exhibits the correct matter spectrum of the MSSM at low energies, but which also contains unrealistic interaction terms. Indeed, in this model the third generation is not hierarchically heavier than the two lighter generations. Moreover, the neutrinos of the Standard Model are exactly massless. Finally, the model contains superpotential terms which lead to rapid proton decay. After explaining the primary features of this model, we next present a more refined example of admissible matter curves which rectifies all of the above issues.

As a first example, consider a model with fractional line bundle $L = \mathcal{O}_S(E_3 - E_4)^{1/5}$ and matter content localized on the following choice of matter curves:

Model I	Curve	Class	g_Σ	L_Σ	L'^{m}_{Σ}
$1 \times \left(5_H + \overline{5}_H\right)$	Σ_H	$-K_S$	1	$\mathcal{O}_{\Sigma_H}(p_1 - p_2)^{1/5}$	$\mathcal{O}_{\Sigma_H}(p_1 - p_2)^{-3/5}$
$3 \times \overline{5}_M$	$\Sigma_M^{(1)}$	E_1	0	$\mathcal{O}_{\Sigma_M^{(1)}}$	$\mathcal{O}_{\Sigma_M^{(1)}}(-3)$
$2 \times 10_M$	$\Sigma_M^{(2)}$	$H - E_1 - E_2$	0	$\mathcal{O}_{\Sigma_M^{(2)}}$	$\mathcal{O}_{\Sigma_M^{(2)}}(2)$
$1 \times 10_M$	$\Sigma_M^{(3)}$	E_2	0	$\mathcal{O}_{\Sigma_M^{(3)}}$	$\mathcal{O}_{\Sigma_M^{(3)}}(1)$

(17.7)

where p_1 and p_2 denote two divisors on Σ_H which are not linearly equivalent and we have indicated how L restricts on each matter curve as well as the gauge bundle content of each GUT multiplet due to the restriction of the line bundle L' on S' to the various matter curves. By construction, we find that a vector-like pair of Higgs doublets localizes on Σ_H. The degree of the line bundles on each of the chiral matter curves has been chosen to reproduce the correct multiplicity in the MSSM.

In terms of SU(5) GUT multiplets, the schematic form of the superpotential is:

$$W_{\mathrm{SU}(5)} = \lambda^d_{ij} \cdot \overline{5}_H \times \overline{5}^{(i)}_M \times 10^{(j)}_M + \lambda^u_j \cdot 5_H \times 10_M \times 10^{(j)}_M \tag{17.8}$$

where $i = 1, 2, 3$ labels the three generations of $\overline{5}_M$ all localized on a single matter curve and $j = 1, 2$ labels the two generations of 10_M localized on the matter curve $\Sigma^{(2)}_M$. More generally, the superpotential may also contain interactions which involve gauge singlets which take the schematic form $1 \times 5 \times \overline{5}$. Such interactions can then lead to a μ term for the Higgs and a Dirac mass term for the neutrinos.

As the above example demonstrates, there are potentially many admissible local models of this type which can all yield the matter content of the MSSM. Although this model possesses non-trivial interaction terms, it is unclear whether these terms are consistent with constraints from low energy physics. As argued in subsection 14.2, when no curves self-intersect or pinch inside of S, the corresponding Yukawa couplings do not produce the correct hierarchy in quark masses. Moreover, as explained in section 12, because H_u and H_d localize on the same matter curve, lifting the Higgs triplets via fluxes can still induce quartic terms in the superpotential of the form $QQQL/M_{KK}$ with order one coefficients. Finally, in addition to an incorrect hierarchy for the quarks, the neutrinos are exactly massless in this model.

We now present a different configuration of matter curves which resolves all of the problems mentioned above. To this end, we require that at least one generation localize on a self-intersecting curve. For simplicity, we place all three generations of 10_M's on a self-intersecting $\mathbb{P}^1$ and all three generations of $\overline{5}_M$'s on a smooth $\mathbb{P}^1$ which does not self-intersect. With the same choice of $L = \mathcal{O}_S(E_3 - E_4)^{1/5}$ as in the previous example, the matter content, line bundle assignments and effective class of each matter curve are:

Model II	Curve	Class	g_Σ	L_Σ	L'^n_Σ
$1 \times 5_H$	$\Sigma^{(u)}_H$	$H - E_1 - E_3$	0	$\mathcal{O}_{\Sigma^{(u)}_H}(1)^{1/5}$	$\mathcal{O}_{\Sigma^{(u)}_H}(1)^{2/5}$
$1 \times \overline{5}_H$	$\Sigma^{(d)}_H$	$H - E_2 - E_4$	0	$\mathcal{O}_{\Sigma^{(d)}_H}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^{(d)}_H}(-1)^{2/5}$
$3 \times 10_M$	$\Sigma^{(1)}_M$ (pinched)	$2H - E_1 - E_5$	0	$\mathcal{O}_{\Sigma^{(1)}_M}$	$\mathcal{O}_{\Sigma^{(1)}_M}(3)$
$3 \times \overline{5}_M$	$\Sigma^{(2)}_M$	H	0	$\mathcal{O}_{\Sigma^{(2)}_M}$	$\mathcal{O}_{\Sigma^{(2)}_M}(3)$

. (17.9)

See figure 7 for a depiction of the various matter curves in this model. In computing the multiplicities on the self-intersecting curve we have neglected all subtleties which could occur based on viewing this curve as a pinched genus one curve because the flux data from the non-compact brane is a free discrete parameter which we can always tune to give the correct number of generations. The superpotential now takes the form:

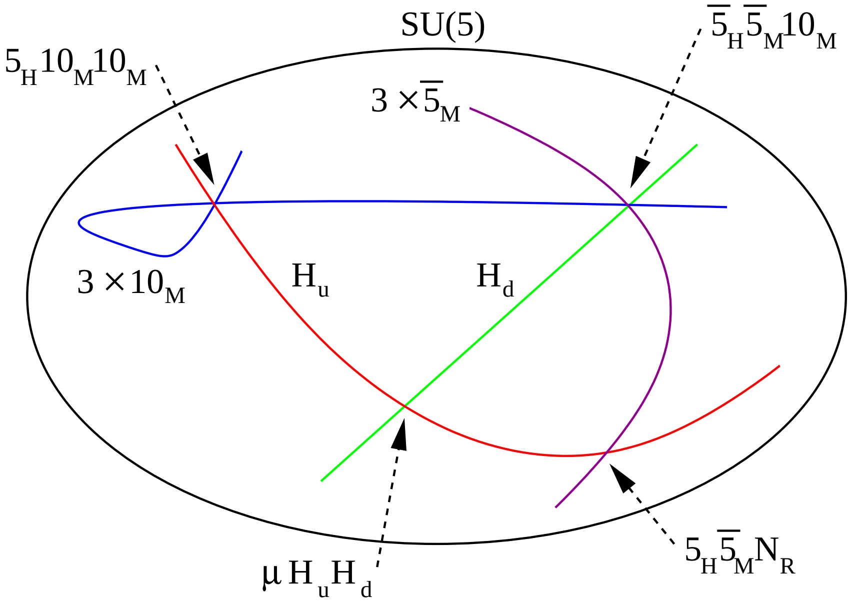

Figure 7: Depiction of the various matter curves in the SU(5) model referred to as "Model II". In this case, all three generations in the 10 of SU(5) localize on one curve and three generations in the $\overline{5}$ localize on another curve. The Higgs up and down curves localize on distinct matter curves and intersect at a point in S. The contributions to the superpotential from the intersection of various matter curves is also indicated.

$$W_{\mathrm{SU}(5)} = \lambda^d_{ij} \cdot \overline{5}_H \times \overline{5}^{(i)}_M \times 10^{(j)}_M + \lambda^u_{ij} \cdot 5_H \times 10^{(i)}_M \times 10^{(j)}_M \tag{17.10}$$

$$+ \rho^{ia}_{\mathrm{repel}} \cdot 5_H \times \overline{5}^{(i)}_M \times N^{(a)}_R + \lambda_{\mathrm{repel}} \cdot \Phi \times 5_H \times \overline{5}_H \tag{17.11}$$

where in the above, the intersection between $\Sigma^{(u)}_H$ and $\Sigma^{(2)}_M$ leads to a two-fold enhancement in rank to an SU(7) singularity so that the singlet $N^{(a)}_R$ may be identified with the right-handed neutrinos and the vev of Φ determines the supersymmetric μ term. In this model, the neutrino masses are purely of Dirac type. As explained in section 15, these gauge singlet wave functions can generate an exponential suppression of the expected type. Finally, as explained in greater detail in section 12, because the H_u and H_d fields localize on distinct matter curves, the operator $QQQL$ is automatically suppressed by a phenomenologically acceptable amount.

18. Evading the no go theorem and flipped models

In the previous sections we have presented many potential ingredients for building models based on $G_S = \mathrm{SU}(5)$. This is partially due to the analysis of subsection 10.2 which shows that for $G_S = \mathrm{SO}(10)$, direct breaking to G_{std} via internal fluxes will always as a byproduct generate exotic matter fields. For surfaces of general type, a partial breaking to $\mathrm{SU}(5) \times \mathrm{U}(1)$ would not present a serious obstruction because after breaking to a four-dimensional GUT group, the remaining breaking can proceed when an adjoint-valued field develops a suitable vev. For del Pezzo models, a similar mechanism exists for flipped GUT models.

We now recall the primary features of four-dimensional flipped SU(5) GUT models [67, 93, 30]. The gauge group of flipped SU(5) is SU(5) $\times$ U(1), which naturally embeds in SO(10). Indeed, the chiral matter content of the Standard Model is given by the flipped SU(5) multiplets:

$$\text{Matter} : 3 \times (1_{-5} + \overline{5}_3 + 10_{-1}) \tag{18.1}$$

$$\text{MSSM Higgs} : 1 \times (5_2 + \overline{5}_{-2}) \tag{18.2}$$

$$\text{GUT Higgs} : 1 \times (10_{-1} + \overline{10}_1) \tag{18.3}$$

where $\mathrm{U}(1)_Y$ of the MSSM corresponds to a linear combination of the U(1) generator in SU(5) and the overall U(1) factor. Due to the fact that the U(1) hypercharge is given by a flipped embedding, the 5_2 contains the Higgs down of the MSSM, while the $\overline{5}_{-2}$ contains the Higgs up. In addition to interaction terms which descend from the $16_M \times 16_M \times 10_H$ in an SO(10) GUT, a flipped SU(5) model includes the interaction terms $5_2 \times 10_{-1} \times 10_{-1}$ and $\overline{5}_{-2} \times \overline{10}_1 \times \overline{10}_1$ between the MSSM and GUT Higgs fields. These interaction terms descend from $16_h \times 16_h \times 10_H$ and $\overline{16}_h \times \overline{16}_h \times 10_H$ in an SO(10) GUT. As explained in [30] there is a unique F- and D-flat direction along which the GUT Higgs 10_{-1} and $\overline{10}_1$ develop a vev. This vev simultaneously breaks SU(5) $\times$ U(1) to SU(3) $\times$ SU(2) $\times$ U(1) while also giving a large mass to the Higgs triplets of the 5_2 and $\overline{5}_{-2}$. In order to emphasize the embedding in SO(10), we shall organize all of the matter content in terms of representations of SO(10). Explicitly, we have:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \tag{18.4}$$

$$16_M = 1_{-5} + \overline{5}_3 + 10_{-1} \tag{18.5}$$

$$10_H = 5_2 + \overline{5}_{-2} \tag{18.6}$$

Because the GUT Higgs fields 10_{-1} and $\overline{10}_1$ do not fill out a complete SO(10) multiplet, we shall refer to these fields as Π and $\overline{\Pi}$, respectively.

We now explain how in F-theory a higher dimensional SO(10) GUT can naturally break to a four-dimensional flipped SU(5) GUT. For concreteness, we consider models based on the del Pezzo 8 surface. The adjoint representation of SO(10) decomposes into representations of SU(5) $\times$ U(1) as:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \tag{18.7}$$

$$45 \to 1_0 + 24_0 + 10_4 + \overline{10}_{-4}. \tag{18.8}$$

By inspection, the U(1) charge assignment of the 10_4 does not correspond to the representation content of any field in a flipped SU(5) model. We therefore require that the zero mode content of the theory must not contain any 10_4's or $\overline{10}_{-4}$'s. In this case, the only gauge bundle configurations which do not contain any such exotics are all of the form $\mathcal{O}_S(\alpha)^{1/4}$ where α corresponds to a simple root of $H_2(S, \mathbb{Z})$.

So long as the instanton configuration breaks G_S to a four-dimensional flipped GUT group with all matter fields in well-defined flipped GUT multiplets, we can avoid additional exotica in the low energy spectrum. For example, in breaking E_6 to $\mathrm{SO}(10) \times \mathrm{U}(1)$, the adjoint decomposes as:

$$E_6 \supset \mathrm{SO}(10) \times \mathrm{U}(1) \tag{18.9}$$

$$78 \rightarrow 1_0 + 45_0 + 16_{-3} + \overline{16}_3. \tag{18.10}$$

Further breaking SO(10) to $\mathrm{SU}(5) \times \mathrm{U}(1)$, if we again require that no zero modes descend from the 45_0 of $\mathrm{SO}(10) \times \mathrm{U}(1)$, we will generically produce zero modes which descend from the 16_{-3} and $\overline{16}_3$. We note that in this case, the zero modes can still organize into complete flipped multiplets.

18.1 Flipped SU(5) model

We now present a hybrid model which partially unifies to a flipped SU(5) GUT as a four-dimensional model and then further unifies to a higher dimensional SO(10) GUT model. Because none of the matter fields of the flipped model descend from the adjoint representation of SO(10), all of the chiral matter content of the flipped SU(5) model must localize on matter curves. Hence, the SO(10) interaction term $16_M \times 16_M \times 10_H$ must originate from the triple intersection of matter curves. To this end, we consider a geometry where the generic SO(10) singularity undergoes a twofold enhancement in rank to E_7 and SO(14) singularities.

Decomposing the adjoint representation of E_7 with respect to the subgroup $\mathrm{SO}(10) \times \mathrm{U}(1) \times \mathrm{U}(1)$ yields:

$$E_7 \supset \mathrm{SO}(10) \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \tag{18.11}$$

$$133 \rightarrow 1_{0,0} + 1_{0,2} + 1_{0,-2} + 1_{0,0} + 45_{0,0} \tag{18.12}$$

$$+ 10_{2,0} + 10_{-2,0} + 16_{-1,1} + 16_{-1,-1} + \overline{16}_{1,1} + \overline{16}_{1,-1} \tag{18.13}$$

so that six-dimensional hypermultiplets in the 16 localize on the two matter curves $(-t_1 + t_2 = 0)$ and $(-t_1 - t_2 = 0)$ and a six-dimensional hypermultiplet in the 10 localizes on the matter curve $(t_1 = 0)$. By inspection, we see that a local enhancement to E_7 can accommodate interaction terms of the form $16 \times 16 \times 10$ and $\overline{16} \times \overline{16} \times 10$. A similar analysis establishes that a local enhancement to SO(14) can accommodate an interaction term of the form $1 \times 10 \times 10$.[24]

We now present a toy hybrid scenario which we refer to as the "Hybrid I" model. Some deficiencies with this example will be rectified in the "Hybrid II" model. The SO(10) GUT group breaks to $\mathrm{SU}(5) \times \mathrm{U}(1)$ with no bulk exotics when the gauge bundle configuration corresponds to the fractional line bundle $L = \mathcal{O}_S(E_1 - E_2)^{1/4}$. In the Hybrid I model, the

[24] In fact, in a previous version of this paper, these local U(1) charge assignments for the explicit flipped models considered were not properly taken into account. We thank J. Marsano, N. Saulina and S. Schäfer-Nameki for bringing this error to our attention.

matter curves and gauge bundle assignments for each curve are:

Hybrid I	Curve	Class	g_Σ	L_Σ	L'^n_Σ
$1 \times 16_M$	$\Sigma_M^{(1)}$	E_3	0	$\mathcal{O}_{\Sigma_M^{(1)}}$	$\mathcal{O}_{\Sigma_M^{(1)}}(1)$
$2 \times 16_M$	$\Sigma_M^{(2)}$	$H - E_3 - E_4$	0	$\mathcal{O}_{\Sigma_M^{(2)}}$	$\mathcal{O}_{\Sigma_M^{(2)}}(2)$
$1 \times 10_H^{(d)}$	$\Sigma_H^{(d)}$	$2H - E_1 - E_3$	0	$\mathcal{O}_{\Sigma_H^{(d)}}(1)^{1/4}$	$\mathcal{O}_{\Sigma_H^{(d)}}(1)^{1/2}$
$1 \times 10_H^{(u)}$	$\Sigma_H^{(u)}$	$2H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_H^{(u)}}(-1)^{1/4}$	$\mathcal{O}_{\Sigma_H^{(u)}}(-1)^{1/2}$
$1 \times (\Pi + \overline{\Pi})$	Σ_h (pinched)	$3H - E_1 - E_2$	1	$\mathcal{O}_{\Sigma_h}(p_1 - p_2)^{1/4}$	$\mathcal{O}_{\Sigma_h}(p_1 - p_2)^{1/4}$

(18.14)

with notation as in (17.7).

By construction, we find one chiral generation of the MSSM localized on $\Sigma_M^{(1)}$ with two generations localized on $\Sigma_M^{(2)}$. The matter curve $\Sigma_H^{(d)}$ supports a zero mode transforming in the representation $5_2^{(d)}$ which contains the Higgs down of a flipped GUT model, and $\Sigma_H^{(u)}$ supports a single zero mode in the $\overline{5}_{-2}^{(u)}$. Finally, in addition to the matter content of the MSSM, we have also included a single vector-like pair of GUT Higgs fields Π and $\overline{\Pi}$.

Including terms up to quartic order, the resulting superpotential of the four-dimensional flipped SU(5) model is therefore:

$$W_{\mathrm{SU}(5)\times\mathrm{U}(1)} = W_{\mathrm{Matter}} + W_{\mathrm{Higgs}} + W_{\mathrm{Quartic}} \tag{18.15}$$

where the interaction terms for the chiral matter are:

$$W_{\mathrm{Matter}} = \lambda_i^u (\overline{5}_{-2}^{(u)} \times \overline{5}_3^{(i)} \times 10_{-1}^{(3)} + \overline{5}_{-2}^{(u)} \times 10_{-1}^{(i)} \times \overline{5}_3^{(i)}) \tag{18.16}$$

$$+ \lambda_i^d (5_2^{(d)} \times 1_{-5}^{(i)} \times \overline{5}_3^{(3)} + 5_2^{(d)} \times 10_{-1}^{(i)} \times 10_{-1}^{(3)}) \tag{18.17}$$

and $i = 1, 2$ runs over two of the generations of the MSSM. The interaction terms in the Higgs sector are:

$$W_{\mathrm{Higgs}} = \lambda_{\mathrm{repel}} \cdot \Phi \times 5_2^{(d)} \times \overline{5}_{-2}^{(u)} + \lambda_\Pi \cdot 5_2^{(d)} \times \Pi \times \Pi + \lambda_{\overline{\Pi}} \cdot \overline{5}_{-2}^{(u)} \times \overline{\Pi} \times \overline{\Pi}. \tag{18.18}$$

The final term W_{Quartic} originates from integrating out the heavy Kaluza-Klein modes associated with the Higgs fields:

$$W_{\mathrm{Quartic}} = \frac{c_i}{M_{KK}} 10_{-1}^{(i)} \times \overline{\Pi} \times 10_{-1}^{(3)} \times \overline{\Pi}. \tag{18.19}$$

In the above, the mass scale M_{KK} is the overall Kaluza-Klein mass scale. In general, this can be slightly higher than the vev of the GUT Higgs fields. We note that when $\overline{\Pi}$ develops a vev which also lifts the Higgs triplets from the low energy spectrum, it also generates a large Majorana mass term for the right-handed neutrinos.

Because the matter curves $\Sigma_M^{(1)}$ and $\Sigma_M^{(2)}$ do not self-intersect, the resulting model has two heavy generations. In contrast to the minimal SU(5) models considered previously, the field-theoretic missing partner mechanism already lifts the Higgs triplets and prevents the higher dimension $QQQL$ operator from being generated. Moreover, the model already incorporates a natural seesaw mechanism.

Before proceeding to a slightly more realistic model, we note that although it would at first appear to be more economical to place the Higgs up and Higgs down on the same matter curve, this leads to certain undesirable consequences in the low energy theory. The reason is that the Higgs up and down fields would then be equally or oppositely charged under a common U(1) symmetry. This would either forbid the coupling $16 \times 16 \times 10$ or $\overline{16} \times \overline{16} \times 10$ in the low energy theory. The former interaction is necessary for generating semi-realistic Yukawa couplings, while the latter is necessary for implementing doublet-triplet splitting using the missing partner mechanism. In order to achieve both couplings, it appears necessary to localize these fields on different matter curves.

A more realistic hierarchy in quark masses can be achieved when the chiral matter curves self-intersect. As a small refinement on the above model, we take $L = \mathcal{O}_S(E_1 - E_2)^{1/4}$ as before, while the matter curves and gauge bundle assignments for each curve are now:

Hybrid II	Curve	Class	g_Σ	L_Σ	$L_\Sigma^{\prime n}$
$1 \times 10_H^{(d)}$	$\Sigma_H^{(d)}$	$2H - E_1 - E_3$	0	$\mathcal{O}_{\Sigma_H^{(d)}}(1)^{1/4}$	$\mathcal{O}_{\Sigma_H^{(d)}}(1)^{1/2}$
$1 \times 10_H^{(u)}$	$\Sigma_H^{(u)}$	$2H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_H^{(u)}}(-1)^{1/4}$	$\mathcal{O}_{\Sigma_H^{(u)}}(-1)^{1/2}$
$3 \times 16_M$	Σ_M (pinched)	$3H$	1	$\mathcal{O}_{\Sigma_M}$	$\mathcal{O}_{\Sigma_M}(3p')$
$1 \times (\Pi + \overline{\Pi})$	Σ_h (pinched)	$3H - E_1 - E_2$	1	$\mathcal{O}_{\Sigma_h}(p_1 - p_2)^{1/4}$	$\mathcal{O}_{\Sigma_h}(p_1 - p_2)^{1/4}$

(18.20)

so that all three generations localize on the matter curve Σ_M. See figure 8 for a depiction of the Hybrid II model. While the zero mode content of this case is the same as the Hybrid I model, the self-intersection of the matter curves allows the model to have one generation which is hierarchically heavier than the lighter two generations, much as in the second minimal SU(5) example of section 17. Aside from this difference, the structure of the superpotential is quite similar to that given by equation (18.15). Indeed, just as in the Hybrid I model, there exist higher dimension operators which can generate large Majorana mass terms for the right-handed neutrinos.

19. Numerology

Throughout this paper we have given numerical estimates of various quantities which appear to be in rough agreement with observation. In this section we demonstrate that for an appropriate choice of order one constants, many of the relations obtained throughout are in agreement with experimental observation. Our point here is not so much to show that we can match to the precise numerical values, but rather that the numbers we have obtained are not wildly different from the expected ranges. Indeed, although we shall typically evaluate all quantities at the GUT scale, in a more accurate analysis these quantities would of course have to be evolved under renormalization group flow to low energies. In this regard, our order of magnitude estimates will be somewhat naive, although we believe it still gives a reliable guide for the ranges of energy scales involved in our models. Moreover, for concreteness, in this section we focus on the case of the minimal SU(5) model.

At the level of precision with which we can reliably estimate parameters, all of our estimates depend on order one coefficients, the Planck mass $M_{\rm pl}$, the GUT scale $M_{\rm GUT}$,

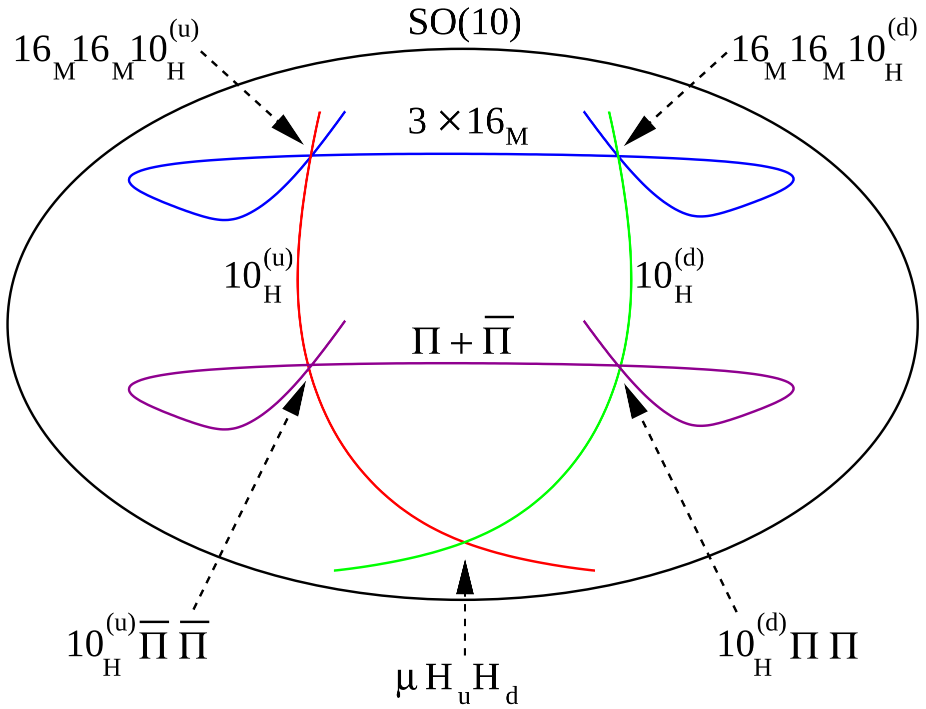

Figure 8: Depiction of the various matter curves in the flipped SU(5) model referred to as "Hybrid II" in the text. The background instanton configuration breaks the bulk gauge group SO(10) to SU(5) × U(1). In this case, all three generations transform in the 16 of SO(10) and localize on a single self-intersecting matter curve. The MSSM Higgs fields descend from two different 10's of SO(10). The model also contains a single vector-like pair transforming in the 10_{-1} and $\overline{10}_{+1}$ of SU(5) × U(1) which facilitates GUT group breaking and doublet-triplet splitting. These GUT Higgs fields descend from a six-dimensional hypermultiplet transforming in the 16 of SO(10).

the Higgs up vev, and the value of the gauge coupling constants at the GUT scale, $\alpha_{\rm GUT}$. Throughout, we use the following approximate values:

$$M_{\rm pl} \sim 1 \times 10^{19} \text{ GeV} \tag{19.1}$$

$$M_{\rm GUT} \sim 3 \times 10^{16} \text{ GeV} \tag{19.2}$$

$$\langle H_u \rangle \sim 246 \text{ GeV} \tag{19.3}$$

$$\alpha_{\rm GUT} = \frac{g^2_{\rm YM}(M_{\rm GUT})}{4\pi} \sim \frac{1}{25}. \tag{19.4}$$

In general, factors of 2 and π are typically beyond the level of precision which we can reliably estimate.

The above parameters appear geometrically as the length scale R_S associated with the size of the del Pezzo, R_B which is associated with the size of the threefold base, and $R_\perp$ which may be viewed as a local cutoff on the behavior of wave functions in the model. These length scales are related by appropriate powers of the small parameter:

$$\varepsilon = \frac{M_{\rm GUT}}{\alpha_{\rm GUT} M_{\rm pl}} \sim 7.5 \times 10^{-2}. \tag{19.5}$$

The various length scales are then given by:

$$\frac{1}{R_S} = M_{\text{GUT}} \sim 3 \times 10^{16} \text{ GeV} \tag{19.6}$$

$$\frac{1}{R_B} = M_{\text{GUT}} \times \varepsilon^{1/3} \sim 1 \times 10^{16} \text{ GeV} \tag{19.7}$$

$$\frac{1}{R_\perp} = M_{\text{GUT}} \times \varepsilon^{\gamma} \sim 5 \times 10^{15\pm 0.5} \text{ GeV} \tag{19.8}$$

where the parameter $1/3 \lesssim \gamma \lesssim 1$ ranges from $1/3$ when B_3 is homogeneous, to 1 when B_3 is given by a tubular geometry.

We now collect and slightly expand on the estimates obtained throughout this paper. We begin by discussing the mass scales associated with quarks. In this case, the masses of the quarks at the GUT scale are very roughly given by:

$$m_q \sim \alpha_{\text{GUT}}^{3/4} \langle H_u \rangle \sim 20 \text{ GeV}. \tag{19.9}$$

Note that the top quark mass is about a factor of 3 higher than this (taking into account the RG flow), which suggests that perhaps the corresponding curves are smaller by that factor to give the correct wave function normalization.

We have also seen that matter fields which localize on curves normal to S in the threefold base B_3 can provide a natural mechanism for generating light neutrino masses as well an exponentially suppressed μ term. As an intermediate case, we have shown that right-handed neutrino wave functions which are attracted to the seven-brane can potentially realize a viable seesaw mechanism. Reproducing equation (15.38) for the convenience of the reader, the light neutrino mass in the seesaw scenario is:

$$m_{\text{light}} \sim \alpha_{\text{GUT}}^{3/4} \frac{\langle H_u \rangle^2}{M_{\text{GUT}}} \varepsilon^{-4\gamma} \sim 2 \times 10^{-1\pm 1.5} \text{ eV}. \tag{19.10}$$

Gauge singlet wave functions can also exhibit more extreme behavior. Indeed, when the Higgs up and down fields localized on different matter curves which intersect, they interact with a gauge singlet wave function outside of S. When this wave function is exponentially suppressed, the induced μ term is given by equation (15.33):

$$\mu(c,\gamma) \sim M_{\text{GUT}} \times \alpha_{\text{GUT}}^{3/4} \varepsilon^{2\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right). \tag{19.11}$$

We find that when c and γ are order one numbers, this value can naturally fall near the electroweak scale. For example, we have:

$$\mu(c = 1/7, \gamma = 1) \sim 140 \text{ GeV} \tag{19.12}$$

$$\mu(c = 1, \gamma = 0.64) \sim 107 \text{ GeV}. \tag{19.13}$$

In a scenario where the neutrinos are purely of Dirac type, an exponentially small value can also be achieved when the gauge singlet wave function is exponentially damped near S. The Dirac mass is given by equation (15.39):

$$m_{\text{Dirac}}(c,\gamma) \sim \langle H_u \rangle \times \alpha_{\text{GUT}}^{3/4} \varepsilon^{\gamma} \exp\left(-\frac{c}{\varepsilon^{2\gamma}}\right). \tag{19.14}$$

As for the μ term, order one values of c and γ yield reasonable values for the masses. Indeed, as explained in subsection 15.5, when the exponential suppression factors are identical for the Dirac neutrino mass and μ term, we obtain the estimate:

$$m_{\mathrm{Dirac}}(c,\gamma) \sim \mu(c,\gamma)\frac{\langle H_u \rangle}{M_{\mathrm{GUT}}} \times \varepsilon^{-\gamma} \sim 0.5 \times 10^{-2\pm 0.5} \text{ eV} \tag{19.15}$$

when $\mu(c,\gamma) \sim 100\,\mathrm{GeV}$. We have also observed that a similar analysis of Yukawa couplings also applies in estimates of the messenger mass scales for gauge mediated supersymmetry breaking scenarios.

20. Conclusions

F-theory provides a natural setup for studying GUT models in string theory. In this paper we have adopted a bottom up approach to string phenomenology and have found that it provides a surprisingly powerful constraint on low energy physics. One's natural expectation is that there should be a great deal of flexibility in local models where issues pertaining to a globally consistent compactification can always be deferred to a later stage of analysis. This is indeed the case in models where a sufficiently loose definition of "local data" is adopted so that gravity need not decouple, and we have given some examples along these lines. Strictly speaking, though, a local model is well-defined by local data when the model admits a limit where it is in principle possible to decouple the GUT scale from the Planck scale. Perhaps surprisingly, this qualitative condition endows these GUT models with considerable predictive power.

The main lesson we have learned is that the mere *existence* of a decoupling limit constrains both the local geometry of the compactification as well as the type of seven-brane which can wrap a compact surface in the local model. To realize a GUT model with no low energy exotics, the bulk gauge group of the seven-brane can only have rank four, five or six, and in order for a decoupling limit to even exist in principle, the seven-brane must wrap a del Pezzo surface. Moreover, all of the vacua which descend at low energies to the MSSM in four dimensions all possess an internal U(1) hypercharge flux on the del Pezzo which at least partially breaks the GUT group. For concreteness, in this paper we have primarily focused on the cases where the bulk gauge group in eight dimensions is SU(5) or SO(10).

In the minimal SU(5) model, all of the matter content at low energies derives from the intersection of the GUT model seven-brane with additional non-compact seven-branes. We have explained how the fields which localize at such intersections can only transform in the 5, 10 or complex conjugate representations. Moreover, the interaction terms are all cubic in the matter fields because the superpotential derives from the triple intersection of matter curves. Matter fields which are neutral under the GUT group localize on matter curves which are orthogonal to the brane. When the gauge singlet is attracted to the seven-brane, the corresponding Yukawa couplings behave as though the gauge singlet had localized inside of S. On the other hand, when the gauge singlet wave function is repelled away from the seven-brane, this can yield a significant exponential suppression in the value of the Yukawa couplings on the order of $\exp(-c/\varepsilon^{2\gamma})$ where c and γ are order one positive numbers and $\varepsilon \sim \alpha_{\rm GUT}^{-1} M_{\rm GUT}/M_{\rm pl}$. In particular, vector-like pairs in such compactifications do not always develop masses on the order of $M_{\rm GUT}$. This runs counter to a coarse effective field theory analysis which would otherwise suggest that such pairs should always develop large masses. In fact, we have seen that this is consistent with a more refined effective field theory analysis because there are typically additional global symmetries present in the low energy theory.

The exponential suppression of such Yukawa couplings naturally solves the μ problem and also provides a natural mechanism for generating acceptably light neutrino masses. The wave function for the right-handed neutrino is either attracted or repelled away from the del Pezzo surface. In the repulsive case, the neutrino mass term is purely of Dirac type and is on the order of $0.5 \times 10^{-2\pm0.5}$ eV. In the attractive case, we find a natural implementation of a modified seesaw mechanism so that the light neutrinos masses are $2 \times 10^{-1\pm1.5}$ eV and the Majorana mass is $\sim 3 \times 10^{12\pm1.5}$ GeV, which is naturally smaller than the simplest GUT seesaw models.

The combination of non-trivial hypercharge flux in the internal dimensions and the existence of additional fluxes derived from the transversally intersecting seven-branes alleviates a number of problems which plague four-dimensional supersymmetric GUT models. The doublet-triplet splitting problem reduces to the condition that the hypercharge flux and flux from the other seven-branes both pierce the Higgs matter curves, while the net hypercharge flux vanishes on curves which support full GUT multiplets.

The internal U(1) hypercharge flux also provides a qualitative explanation for why the $b-\tau$ GUT mass relation approximately matches with observation while the lighter two generations at best obey distorted versions of this relation. This is in a sense the remnant of the mechanism that solves the doublet-triplet splitting problem. Even though the net hypercharge flux vanishes on a matter curve which supports a complete GUT multiplet, the field strength is not identically zero. In this way, the GUT multiplet wave functions experience an Aharanov-Bohm effect which increasingly distorts the GUT mass relations as the mass of the GUT multiplet decreases. In fact, this mechanism *requires* that the internal hypercharge gauge field be non-trivial.

This flux will also typically generate a threshold correction to the unification of the gauge couplings. While there are potentially many other such threshold corrections due to Kaluza-Klein modes, it would clearly be of interest to see whether at least some of these corrections can be reliably estimated in our setup.

The geometry of the compactification can also prevent the proton from decaying too rapidly. Cubic terms in the superpotential are typically excluded in a bottom up approach by requiring that the theory is invariant under R-parity. We have found two ways that the geometry can forbid the same interaction terms which R-parity removes. In one case, R-parity corresponds to a suitable $\mathbb{Z}_2$ symmetry in the geometry of the Calabi-Yau fourfold. At a topological level, the absence of R-parity violating cubic interaction terms corresponds to a technically natural restriction on which matter curves intersect. In the scenario where R-parity descends from a $\mathbb{Z}_2$ group action on the Calabi-Yau, the hypercharge flux and the Higgs matter curves are invariant under this group action while the matter curves are odd. Due to the $\mathbb{Z}_2$ symmetry, the net hypercharge flux must vanish on matter curves which are odd under this group action. Hence, this *automatically forces* the localized matter to organize in complete GUT multiplets. Note that this symmetry also permits a non-vanishing hypercharge flux on the Higgs curves, which is consistent with our solution to the doublet-triplet splitting problem. At higher order in the effective superpotential, the topological condition determining which curves intersect also forbids potentially dangerous baryon number violating quartic operators in the superpotential. Indeed, placing the Higgs up and down fields on distinct matter curves equips the matter fields with additional global symmetries which can forbid such operators.

We have also shown how the geometry of the matter curves translates in the low energy effective theory into non-trivial structure in the Yukawa couplings. The coarsest features of textures follow from the discrete data determining how matter curves intersect inside the seven-brane so that texture zeroes are generically present. We have also presented some speculations on potential ways that additional structure in the Yukawa couplings could arise from a geometrical realization of the Froggatt-Nielsen mechanism, or through an interpretation of the discrete automorphism group of a del Pezzo surface as a flavor group symmetry.

Communicating supersymmetry breaking is also straightforward in this setup. Indeed, we have shown that the geometry of del Pezzo surfaces can easily accommodate vector-like pairs of GUT multiplets localized on isolated matter curves. These vector-like pairs can then serve as the messenger fields in gauge mediated supersymmetry breaking. We have presented different scenarios showing the flexibility of this approach. Depending on the case at hand, the messenger masses can range from near the GUT scale, to energy scales which are significantly lower. Moreover, because we have an independent mechanism for naturally suppressing the μ term, this class of models preserves the best features of gauge mediation models while avoiding the notoriously difficult issue of generating μ and $B\mu$ at around the electroweak scale.

It is perhaps surprising that a few key ideas seem to resolve many problems simultaneously. Indeed, the overall economy in these ingredients lends substantial credence to the basic framework. On the other hand, it is also clear that we have by no means exhausted the potential avenues of investigation. A more systematic study of textures and choices of matter curves, as well as the geometric underpinning of the corresponding Calabi-Yau fourfold are all issues which deserve further attention. In addition, the communication of supersymmetry breaking is simple enough in our setup that it could potentially lead to observable predictions at the LHC. It would clearly be of interest to study such a scenario further.

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A. Gauge theory anomalies and seven-branes

In this appendix we further elaborate on the geometric condition for the low energy spectrum to be free of gauge theory anomalies. First recall the well-studied case of perturbatively realized gauge theories obtained as the low energy limit of D-brane probes of non-compact Calabi-Yau singularities. The condition that all gauge theory anomalies must cancel is equivalent to the requirement that in a consistent bound state of D3-, D5- and D7-branes, the total RR flux measured over a compact cycle must vanish [94]. Even in a non-compact Calabi-Yau threefold given by the total space $\mathcal{O}(K_S) \to S$ with S a Kähler surface, the theory of a stack of D7-branes wrapping S is inconsistent because the self-intersection of the divisor S in the Calabi-Yau threefold is a compact Riemann surface. In a globally consistent model, additional O7-planes must be introduced to cancel the corresponding RR tadpole. Indeed, a consistent compactification of F-theory on an elliptically fibered Calabi-Yau fourfold will automatically contain similar contributions so that the net monodromy around all seven-branes is trivial.

Next consider the potential contribution from D5-branes to a candidate bound state. Letting $[\Sigma_{D5}]$ denote the total homology class of D5-branes wrapping compact two-cycles in $H_2(S,\mathbb{Z})$, the resulting theory is consistent provided:

$$[\Sigma_{D5}] \cdot K_S = 0. \tag{A.1}$$

There is no analogous condition for D3-branes in a non-compact model because the flux lines can escape to infinity in the non-compact model.

In this appendix we consider more general intersecting seven-brane configurations with chiral matter induced from a non-trivial field strength. Using the fact that a low energy theory must be free of non-abelian gauge anomalies, we determine the geometric analogue of equation (A.1) for intersecting $A \times A$ and $D \times A$ brane configurations in a broader class of F-theory compactifications. We also present an example of anomaly cancelation for an E_7 exceptional brane theory.

A.1 $A \times A$ anomalies

We now consider seven-branes wrapping two Kähler surfaces S and S' such that the gauge group of the respective seven-branes is $G_S = \mathrm{SU}(N)$ and $G_{S'} = \mathrm{SU}(N')$ with a six-dimensional bifundamental localized along a matter curve $\Sigma = S \cap S'$. Because only instanton configurations with an overall U(1) factor can induce chirality in the bulk and on matter curves, it is enough to consider instanton configurations in S and S' taking values in $\mathrm{U}(1)^n$ and $\mathrm{U}(1)^{n'}$ for some $n \leq N-1$ and $n' \leq N'-1$.

We consider a breaking pattern such that $\mathrm{SU}(N)$ decomposes into non-abelian subgroup factors $\mathrm{SU}(N_1), \ldots, \mathrm{SU}(N_n)$. Similar conventions will also hold for the decomposition of the gauge group $\mathrm{SU}(N')$. Letting $\overrightarrow{q}$ denote the charge of a representation under the $\mathrm{U}(1)^{n-1}$ subgroup, the fundamental and adjoint representation decompose as:

$$\mathrm{SU}(N_1 + \ldots + N_n) \supset \mathrm{SU}(N_1) \times \ldots \times \mathrm{SU}(N_n) \times \mathrm{U}(1)^{n-1} \tag{A.2}$$

$$N \to (N_1)_{\overrightarrow{q}_1} \oplus \ldots . \oplus (N_n)_{\overrightarrow{q}_n} \tag{A.3}$$

$$A_N \to (A_{N_1})_{2\overrightarrow{q}_1} \oplus \ldots . \oplus (A_{N_n})_{2\overrightarrow{q}_n} \tag{A.4}$$

$$\oplus \left[\underset{i<j}{\oplus} (N_i \times N_j)_{\overrightarrow{q}_i + \overrightarrow{q}_j} \right] \tag{A.5}$$

$$ad(\mathrm{SU}(N)) \to \overset{n}{\underset{i=1}{\oplus}} ad(\mathrm{SU}(N_i))_0 \oplus \left[\underset{i\neq j}{\oplus} (N_i \times \overline{N}_j)_{\overrightarrow{q}_i - \overrightarrow{q}_j} \right] \tag{A.6}$$

where for future use we have also indicated how the two index anti-symmetric representation A_N decomposes. In the above, the charge assignments $\overrightarrow{q}_i$ satisfy the tracelessness condition:

$$\sum_{i=1}^{n} N_i \overrightarrow{q}_i = 0 \tag{A.7}$$

Letting $L_1, \ldots, L_{n-1}$ denote the line bundles which determine the $\mathrm{U}(1)^{n-1}$ gauge field configuration with similar conventions for L'_i, the chiral matter content transforming in the fundamental representation N_i of $\mathrm{SU}(N_i)$ in S and Σ are given by the indices derived in [15]:

$$\#(N_i \times \overline{N}_j)_{\overrightarrow{q}_i - \overrightarrow{q}_j} = -\, c_1(S) \cdot c_1 \left(L_1^{\pi_1(\overrightarrow{q}_i - \overrightarrow{q}_j)} \right) + \ldots \tag{A.8}$$

$$+ -c_1(S) \cdot c_1 \left(L_{n-1}^{\pi_{n-1}(\overrightarrow{q}_i - \overrightarrow{q}_j)} \right) \tag{A.9}$$

$$\#(N_i)_{\overrightarrow{q}_i} \times (N'_{i'})_{\overrightarrow{q}'_{i'}} = \deg L_{1|\Sigma}^{\pi_1(\overrightarrow{q}_i)} + \ldots + \deg L_{n|\Sigma}^{\pi_n(\overrightarrow{q}_i)} \tag{A.10}$$

$$+ \deg L_{1|\Sigma}^{\prime \pi'_1(\overrightarrow{q}'_{i'})} + \ldots + \deg L_{n'|\Sigma}^{\prime \pi'_n(\overrightarrow{q}'_{i'})} \tag{A.11}$$

where π_i denotes the projection onto the i^{th} component of a given charge vector, and a negative number indicates the net chiral matter content transforms in the complex conjugate representation.

The net anomaly coefficient a_i of the $\mathrm{SU}(N_i)$ factor is given by summing over all contributions to the fundamental representation of $\mathrm{SU}(N_i)$. Letting $d_i = c_1(S) \cdot c_1(L_i)$ and $d_{i|\Sigma} = \deg L_{i|\Sigma}$, we find:

$$a_i = -\sum_{j=1}^{n} N_j \sum_{k=1}^{n-1} \pi_k(\overrightarrow{q}_i - \overrightarrow{q}_j) d_k \tag{A.12}$$

$$+ \sum_{i'=1}^{n'} N'_{i'} \left(\sum_{k=1}^{n-1} \pi_k(\overrightarrow{q}_i) d_{k|\Sigma} + \sum_{k'=1}^{n'-1} \pi'_{k'}(\overrightarrow{q}'_{i'}) d'_{k'|\Sigma} \right). \tag{A.13}$$

Simplifying this expression using the tracelessness condition of equation (A.7) and the analogous condition for the $N_{i'}$ now implies:

$$a_i = -N \sum_{k=1}^{n-1} \pi_k(\overrightarrow{q}_i) d_k + N' \sum_{k=1}^{n-1} \pi_k(\overrightarrow{q}_i) d_{k|\Sigma} \tag{A.14}$$

$$= \int_S c_1(L_1^{\pi_1(\overrightarrow{q}_i)} \otimes \ldots \otimes L_{n-1}^{\pi_{n-1}(\overrightarrow{q}_i)}) c_1 \left(\mathcal{O}_S(K_S)^N \otimes \mathcal{O}_S(\Sigma)^{N'} \right). \tag{A.15}$$

The condition for a_i to vanish is the direct analogue of the perturbatively realized condition in equation (A.1).

A.2 $A \times D$ anomalies

We now consider seven-branes wrapping two Kähler surfaces S and S' such that the gauge group of the respective seven-branes is $G_S = \mathrm{SU}(N)$ and $G_{S'} = \mathrm{SO}(2R+2M)$ with six-dimensional matter fields localized along the curve $\Sigma = S \cap S'$. Decomposing $\mathrm{SO}(2N + 2R+2M) \supset \mathrm{SU}(N) \times \mathrm{SO}(2R+2M) \times \mathrm{U}(1)$, the six-dimensional fields localized on Σ now transform in the representation $(A_N, 1) \oplus (N, 2R)$ of $\mathrm{SU}(N) \times \mathrm{SO}(2R+2M)$. As before, it is enough to treat instanton configurations taking values in the subgroups $\mathrm{U}(1)^n \subset \mathrm{SU}(N)$ and $\mathrm{U}(1)^t \subset \mathrm{SO}(2R)$. In order to simplify the combinatorics associated with breaking patterns of the SO gauge group factor, we confine our analysis to the breaking pattern $\mathrm{SO}(2R+2M) \supset \mathrm{SO}(2R) \times \mathrm{SU}(M) \times \mathrm{U}(1)$. The fundamental and adjoint representations of $\mathrm{SO}(2R+2M)$ decompose into the commutant subgroup of $\mathrm{U}(1)$ as:

$$\mathrm{SO}(2R+2M) \supset \mathrm{SO}(2R) \times \mathrm{SU}(M) \times \mathrm{U}(1) \tag{A.16}$$

$$2R \to (2R)_0 \oplus \left((M)_p \oplus (\overline{M}_i)_{-p}\right) \tag{A.17}$$

$$ad(\mathrm{SO}(2R)) \to 1_0 \oplus ad(\mathrm{SO}(2R_i))_0 \oplus ad(\mathrm{SU}(M_i))_0 \tag{A.18}$$

$$\oplus (A_M)_{2p} \oplus (\overline{A_M})_{-2p} \oplus (2R, M)_p \oplus (2R, \overline{M})_{-p}. \tag{A.19}$$

Consider first non-abelian anomalies associated to the gauge group factor $\mathrm{SU}(N_i)$. In this case, we recall that in a normalization of group generators where the fundamental has anomaly coefficient $+1$, the two index anti-symmetric representation has anomaly coefficient $N_i - 4$. Repeating a similar analysis to that given in the previous section, the total anomaly coefficient for the non-abelian group $\mathrm{SU}(N_i)$ is:

$$a_i = -N\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i)d_k + (2R+2M)\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i)d_{k|\Sigma} \tag{A.20}$$

$$+ (N_i - 4)\sum_{k=1}^{n-1}\pi_k(2\overrightarrow{q}_i)d_{k|\Sigma} + \sum_{j\neq i}N_j\left(\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i+\overrightarrow{q}_j)d_{k|\Sigma}\right) \tag{A.21}$$

$$= -2N\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i)d_k + (2R+2M)\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i)d_{k|\Sigma} - 8\sum_{k=1}^{n-1}\pi_k(\overrightarrow{q}_i)d_{k|\Sigma} \tag{A.22}$$

$$= 2\int_S c_1(L_1^{\pi_1(\overrightarrow{q}_i)} \otimes \ldots \otimes L_{n-1}^{\pi_{n-1}(\overrightarrow{q}_i)})c_1\left(\mathcal{O}_S(K_S)^N \otimes \mathcal{O}_S(\Sigma)^{R+M-4}\right). \tag{A.23}$$

Comparing equations (A.15) and (A.23), the shift $R+M \to R+M-4$ indicates the presence of an $O7$-plane.

Next consider the anomaly coefficient of the $\mathrm{SU}(M)$ factor. In this case, the total anomaly coefficient for the non-abelian group $\mathrm{SU}(M)$ is:

$$b_i = -2p(M-4)d' - 2Rpd' + 2pNd'_\Sigma \tag{A.24}$$

$$= 2\int_{S'} c_1(L'^p)c_1\left(\mathcal{O}_{S'}(\Sigma)^N \otimes \mathcal{O}_{S'}(K_{S'})^{R+M-4}\right). \tag{A.25}$$

Proceeding by induction, it now follows that a similar result also holds for the more general breaking pattern where each $\mathrm{SU}(M)$ and $\mathrm{SO}(2R)$ factor decomposes further.

A.3 E_7 anomalies

The analysis of the previous subsections demonstrates that for A- and D- type seven-branes, the geometric condition for anomaly cancelation in the four-dimensional effective theory relates the total matter content in the bulk with that localized on matter curves. We now determine the analogous condition for a seven-brane with gauge group $G_S = E_7$ and M copies of the 56 localized on a curve Σ. We consider a U(1) gauge field configuration which breaks E_7 to $SU(7) \times U(1)$. The representation content of E_7 decomposes as:

$$E_7 \supset SU(8) \supset SU(7) \times U(1) \tag{A.26}$$

$$56 \to 7_{-6} + \overline{7}_6 + 21_2 + \overline{21}_{-2} \tag{A.27}$$

$$133 \to 1_0 + 7_8 + \overline{7}_{-8} + 48_0 + \overline{35}_4 + 35_{-4} \tag{A.28}$$

where the 21, 35 and $\overline{35}$ denote the two, three and four index anti-symmetric representations of SU(7). It now follows that the chiral matter content derived from S and Σ is:

$$\#7_8 = -c_1(S) \cdot c_1(L^8) \tag{A.29}$$

$$\#35 = -c_1(S) \cdot c_1(L^{-4}) \tag{A.30}$$

$$\#7_{-6} = M \deg L^{-6}_{|\Sigma} \tag{A.31}$$

$$\#21 = M \deg L^{2}_{|\Sigma}. \tag{A.32}$$

To compute the anomaly of the SU(7) theory, we first recall that the anomaly coefficient for the i-index anti-symmetric representation $A_k^{(i)}$ of SU(n) in $2(k-1)$ dimensions is [95]:

$$A_k^{(2)} = n - 2^{k-1} \tag{A.33}$$

$$A_k^{(3)} = \frac{1}{2}n^2 - \frac{1}{2}n(2^k + 1) + 3^{k-1} \tag{A.34}$$

$$A_k^{(4)} = \frac{1}{12}(2n^3 - 3n^2(2^k + 2) + n(4 \times 3^k + 3 \times 2^k + 4) - 3 \times 4^k) \tag{A.35}$$

so that in four dimensions, the anomaly coefficients of the SU(7) theory are $A_k^{(2)} = 3$, $A_k^{(3)} = 2$, $A_k^{(4)} = -2$. Returning to equations (A.29)–(A.32), we note that the contribution to the total anomaly from S and Σ separately cancel in this particular case so that we do not deduce an analogue of equation (A.1).

B. Hypersurfaces in $\mathbb{P}^3$

In this section we review some properties of degree n hypersurfaces H_n in $\mathbb{P}^3$. Further details can be found for example in [96]. Letting H denote the hyperplane class of $\mathbb{P}^3$, the total Chern class of H_n is given by the adjunction formula:

$$c(H_n) = \frac{c(\mathbb{P}^3)}{c(N_{H_n/\mathbb{P}^3})} = 1 + (4-n)H + (6 - 4n + n^2)H^2. \tag{B.1}$$

It thus follows that the Euler character $e(H_n)$, holomorphic Euler characteristic $\chi(\mathcal{O}_{H_n})$ and signature $\tau(H_n)$ are:

$$e(H_n) = \int_{H_n} c_2(H_n) = \int_{\mathbb{P}^3} n(6-4n+n^2)H^3 = n^3 - 4n^2 + 6n \tag{B.2}$$

$$\chi(\mathcal{O}_{H_n}) = \int_{H_n} \frac{c_1(H_n)^2 + c_2(H_n)}{12} = \frac{1}{6}(n^3 - 6n^2 + 11n) \tag{B.3}$$

$$\tau(H_n) = \int_{H_n} \frac{c_1(H_n)^2 - 2c_2(H_n)}{3} = -\frac{1}{3}(n^3 - 4n). \tag{B.4}$$

We next determine the Hodge numbers of H_n. Using the Lefschetz hyperplane theorem, $h^{1,0}(\mathbb{P}^3) = 0$ implies $h^{1,0}(H_n) = 0$. Moreover, because $e(H_n) = 2 + 2h^{2,0} + h^{1,1}$ and $\chi(\mathcal{O}_{H_n}) = 1 - h^{0,1} + h^{0,2} = 1 + h^{0,2}$, equations (B.2)–(B.4) imply:

$$h^{1,1}(H_n) = \frac{1}{3}(2n^3 - 6n^2 + 7n) \tag{B.5}$$

$$h^{2,0}(H_n) = \frac{1}{6}(n^3 - 6n^2 + 11n) - 1 \tag{B.6}$$

$$b_2(H_n) = n^3 - 4n^2 + 6n - 2. \tag{B.7}$$

The last expression determines the dimension of $H_2(H_n, \mathbb{Z})$ as a lattice over the integers. It follows from Poincaré duality that when equipped with the intersection pairing of the geometry, this lattice is self-dual. Moreover, returning to equation (B.1), reduction of $c_1(H_n)$ mod 2 implies that H_n is spin when n is even.[25] This in turn implies that the lattice $H_2(H_n, \mathbb{Z})$ is even (resp. odd) for n even (resp. odd). Because the signature and dimension uniquely determine a lattice with indefinite signature, we conclude that the lattice is of the general form:

$$H_2(H_n, \mathbb{Z}) \simeq (+1)^{\oplus(b_2+\tau)/2} \oplus (-1)^{\oplus(b_2-\tau)/2} \quad (n \text{ odd}) \tag{B.8}$$

$$H_2(H_n, \mathbb{Z}) \simeq (-E_8)^{\tau/8} \oplus U^{\oplus(b_2-\tau)/2} \quad (n \text{ even}) \tag{B.9}$$

where $-E_8$ is minus the Cartan matrix for E_8 and U is the "hyperbolic element" with entries specified by the Pauli matrix σ_x. The canonical class has self intersection number:

$$K_{H_n} \cdot K_{H_n} = \int_{H_n} c_1(H_n)^2 = n(n-4)^2. \tag{B.10}$$

For many purposes, it is of practical use to have a large number of contractible rational curves inside of a given surface which can serve as matter curves for a given model. We note, however, that general results from the mathematics literature [97, 98] demonstrate that for a generic hypersurface of degree at least five, the minimal genus of a curve is at least two. Indeed, typically a given homology class only corresponds to a holomorphic curve for a specific choice of complex structure. To avoid such subtleties, we consider the

[25] This follows from Wu's theorem and the fact that H_n is simply connected.

blowup of a degree n hypersurface at k points, B_kH_n. While the value of $h^{2,0}$ remains invariant under this process, the canonical class of the resulting space is now given by:

$$K_{B_kH_n} = K_{H_n} + E_1 + \ldots + E_k. \tag{B.11}$$

where the E_i denote the effective classes associated with blown up rational curves.

C. Classification of breaking patterns

In this appendix we classify all possible breaking patterns via instantons for a theory defined by a seven-brane filling $\mathbb{R}^{3,1} \times S$ with bulk gauge group G_S such that the resulting spectrum can in principle contain the matter content of the Standard Model. While breaking patterns for GUT groups is certainly a well-studied topic in the phenomenology literature, as far as we are aware, this question has not been studied from the perspective of F-theory. Indeed, although much of our analysis in this paper has focussed on the cases where the bulk gauge group is SU(5) or SO(10), it seems of use for future potential efforts in this direction to catalogue a broader class of candidate breaking patterns which could in principle arise from compactifications of F-theory. We note that by appealing to gauge invariance and certain basic phenomenological requirements, a partial classification of candidate breaking patterns which can appear in string theory has been given in [99].

Throughout our analysis, we shall assume that our model is generic in the sense that along complex codimension one and two subspaces, the rank of the singularity type can enhance by one or two. While in this paper we have focussed on a minimal class of models where the bulk gauge group is $G_S = \mathrm{SU}(5)$ or $\mathrm{SO}(10)$, there are additional possibilities at higher rank. For example, in higher rank cases it may be possible to allow some of the matter fields of the MSSM to originate from bulk zero modes. We now proceed to an analysis of all possible breaking patterns via instantons which can accommodate the matter content of the MSSM. The relevant group theory material on the decomposition of various irreducible representations may be found in [100, 101].

In keeping with our general philosophy, we shall also assume that the group corresponding to the rank two enhancement in singularity type is a subgroup of E_8. For this reason, the rank of the singularity type can be at most six. Moreover, because the Standard Model gauge group has rank four, it is enough to classify breaking patterns associated with singularities of rank four, five and six. The relevant ADE-type of the singularities are therefore:

$$\text{Rank 4:}\quad A_4,\, D_4 \tag{C.1}$$

$$\text{Rank 5:}\quad A_5,\, D_5 \tag{C.2}$$

$$\text{Rank 6:}\quad A_6,\, D_6,\, E_6. \tag{C.3}$$

The singularity type does not fully determine the gauge group G_S. When the collapsed cycles of the singularity type are permuted under a monodromy in the fiber direction, the resulting gauge group is given by the quotient of the original simply laced group by an outer automorphism. In this way, we can also obtain all non-simply laced groups such as SO($2n+1$), USp($2n$), F_4 and G_2. In what follows we adopt the convention USp(2) $\simeq$ SU(2). It therefore follows that we must analyze the breaking patterns for the following possibilities:

$$\text{Rank 4:}\quad \text{SU}(5),\ \text{SO}(8),\ \text{SO}(9),\ F_4 \tag{C.4}$$

$$\text{Rank 5:}\quad \text{SU}(6),\ \text{SO}(10),\ \text{SO}(11) \tag{C.5}$$

$$\text{Rank 6:}\quad \text{SU}(7),\ \text{SO}(12),\ E_6. \tag{C.6}$$

Note in particular that the bulk gauge group is never of USp type. There are in general many possible ways in which the Standard Model gauge group can embed in the above gauge groups. To classify admissible breaking patterns to the Standard Model gauge group, we shall require that all of the matter content of the Standard Model must be present. While much of our analysis will hold for non-supersymmetric theories as well, we shall typically focus on the field content and interactions of the MSSM. In terms of the gauge group SU(3) $\times$ SU(2) $\times$ U(1), the representation content of the fields of the MSSM are:

Q	U	D	H_d, L	E	H_u
$(3,2)_1$	$(\overline{3},1)_{-4}$	$(\overline{3},1)_2$	$(1,2)_{-3}$	$(1,1)_6$	$(1,2)_3$

. (C.7)

In addition, any realistic model must allow the three superpotential terms:

$$W \supset QUH_u + QDH_d + ELH_d. \tag{C.8}$$

Starting from representations which descend from the decomposition of the adjoint representation of E_8, our strategy will be to rule out as many possible breaking patterns as possible because the representation content is incorrect, or because gauge invariance in the parent theory forbids a required superpotential term. For SO gauge groups, we assume the matter organizes into the fundamental, spinor or adjoint representations. For SU gauge groups, we assume that in addition to the adjoint representation, the matter organizes into one, two or three index anti-symmetric representations.

To classify the possible breaking patterns of a given bulk gauge group G_S, we first list all maximal subgroups. Next, we determine all maximal subgroups of each such subgroup and proceed iteratively until we arrive at the Standard Model gauge group. We note that even for a unique nested sequence of subgroups, there may be several distinct subgroups whose commutant contains the Standard Model gauge group. The classification of these possible subgroups is aided by the fact that the gauge group of the Standard Model has rank four so that the corresponding instanton configuration can only take values in a rank one or two subgroup of a given bulk gauge group G_S.

Although they cannot serve as a bulk gauge group, it is also convenient to list all maximal subgroups of some common lower rank groups which appear frequently. The maximal subgroups of SO(7), SU(4), USp(6), USp(4) and G_2 are:

$$\text{SO}(7) \supset \text{SU}(4) \tag{C.9}$$

$$\text{SO}(7) \supset \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \tag{C.10}$$

$$\mathrm{SO}(7) \supset \mathrm{USp}(4) \times \mathrm{U}(1) \tag{C.11}$$
$$\mathrm{SO}(7) \supset G_2 \tag{C.12}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.13}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.14}$$
$$\mathrm{SU}(4) \supset \mathrm{USp}(4) \tag{C.15}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.16}$$
$$\mathrm{USp}(6) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.17}$$
$$\mathrm{USp}(6) \supset \mathrm{SU}(2) \times \mathrm{USp}(4) \tag{C.18}$$
$$\mathrm{USp}(6) \supset \mathrm{SU}(2) \tag{C.19}$$
$$\mathrm{USp}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.20}$$
$$\mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.21}$$
$$\mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.22}$$
$$\mathrm{USp}(4) \supset \mathrm{SU}(2) \tag{C.23}$$
$$G_2 \supset \mathrm{SU}(3) \tag{C.24}$$
$$G_2 \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.25}$$
$$G_2 \supset \mathrm{SU}(2). \tag{C.26}$$

In the remainder of this appendix, we classify possible breaking patterns via instantons of the bulk gauge group. To further specify the order of breaking in a nested sequence of subgroups, we shall sometimes enclose separate subgroup factors in square brackets.

C.1 Rank four

We now classify all breaking patterns of rank four groups. Although SU(5) is the only group of line (C.4) which contains complex representations, for our higher dimensional theories, it is a priori possible that a suitable U(1) field strength in either the compact or non-compact directions of an intersecting seven-brane theory can induce a net chirality in the resulting gauge group. In the rank four case we list all maximal subgroups even if they do not contain the Standard Model gauge group. This is done because for the higher rank cases, such breaking patterns may become available. In the rank four case, we find that only $G_S = \mathrm{SU}(5)$ is a viable possibility.

C.1.1 SU(5)

There is a single maximal subgroup of SU(5) which contains the Standard Model gauge group. Indeed, the representation content is given by the Georgi-Glashow model:

$$\mathrm{SU}(5) \supset \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \equiv G_{\mathrm{std}} \tag{C.27}$$
$$5 \to (1,2)_3 + (3,1)_{-2} \tag{C.28}$$
$$10 \to (1,1)_6 + (\overline{3},1)_{-4} + (3,2)_1 \tag{C.29}$$
$$24 \to (1,1)_0 + (1,3)_0 + (3,2)_{-5} + (\overline{3},2)_5 + (8,1)_0. \tag{C.30}$$

By turning on an instanton in $\mathrm{U}(1)_Y$, we break to the desired gauge group.

C.1.2 SO(8)

We now proceed to the case of SO(8). The maximal subgroups of SO(8) are [100]:

$$\mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.31}$$
$$\mathrm{SO}(8) \supset \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.32}$$
$$\mathrm{SO}(8) \supset \mathrm{SU}(3) \tag{C.33}$$
$$\mathrm{SO}(8) \supset \mathrm{SO}(7) \tag{C.34}$$
$$\mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{USp}(4). \tag{C.35}$$

Returning to lines (C.9)–(C.26), it follows that there does not exist a breaking pattern which yields G_{std}.

C.1.3 SO(9)

The maximal subgroups of SO(9) are:

$$\mathrm{SO}(9) \supset \mathrm{SO}(8) \tag{C.36}$$
$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(4) \tag{C.37}$$
$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \tag{C.38}$$
$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \tag{C.39}$$
$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{C.40}$$

Of the above possibilities, only line (C.38) contains G_{std}. Breaking to G_{std} via a U(1) instanton yields:

$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \tag{C.41}$$
$$9 \to (3,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.42}$$
$$16 \to (2,1)_3 + (2,3)_{-1} + (2,1)_{-3} + (2,\overline{3})_{+1} \tag{C.43}$$
$$36 \to (3,1)_0 + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 + (3,3)_2 + (3,\overline{3})_{-2}. \tag{C.44}$$

By inspection, all singlets of $\mathrm{SU}(2) \times \mathrm{SU}(3)$ are also neutral under the U(1) factor. It thus follows that SO(9) is ruled out as a candidate.

C.1.4 F_4

The maximal subgroups of F_4 are:

$$F_4 \supset \mathrm{SO}(9) \tag{C.45}$$
$$F_4 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \tag{C.46}$$
$$F_4 \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \tag{C.47}$$
$$F_4 \supset \mathrm{SU}(2) \tag{C.48}$$
$$F_4 \supset \mathrm{SU}(2) \times G_2 \tag{C.49}$$

the first case is excluded by the previous analysis of SO(9), leaving only lines (C.46) and (C.47).

First consider the breaking pattern of (C.46):

$$F_4 \supset \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \tag{C.50}$$
$$26 \to (8,1) + (3,3) + (\overline{3},\overline{3}) \tag{C.51}$$
$$52 \to (8,1) + (1,8) + (6,\overline{3}) + (\overline{6},3). \tag{C.52}$$

Breaking either factor of SU(3) $\supset$ SU(2) $\times$ U(1) via a U(1) instanton, we note that all resulting SU(3) $\times$ SU(2) singlets are also neutral under U(1). We therefore conclude that the breaking pattern of line (C.46) is also excluded.

Next consider the remaining breaking pattern of (C.47) which can descend to the Standard Model gauge group:

$$F_4 \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.53}$$
$$26 \to (2,3)_1 + (2,\overline{3})_{-1} + (2,3)_{-2} + (2,\overline{3})_2 + (2,8)_0 \tag{C.54}$$
$$52 \to (3,1)_0 + (1,1)_0 + (1,6)_2 + (1,\overline{6})_{-2} + (1,8)_0 \tag{C.55}$$
$$+ (2,1)_3 + (2,1)_{-3} + (2,6)_{-1} + (2,\overline{6})_1. \tag{C.56}$$

As before, the resulting singlets of the non-abelian factor are also neutral under the U(1) factor. Summarizing, we find that the only available rank four bulk gauge group which can contain the Standard Model is SU(5).

C.2 Rank five

We now proceed to rank five bulk gauge groups. While it is in principle possible that an SU(2) instanton configuration could produce a consistent breaking pattern to the particle content of the Standard Model, we find that in all cases, the relevant breaking pattern is again always an instanton configuration with structure group U(1) or U(1) $\times$ U(1).

C.2.1 SU(6)

We assume that the matter content organizes into the representations 6, 15, 20 and 35 of SU(6), as well as their dual representations. The maximal subgroups of SU(6) are:

$$\mathrm{SU}(6) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \tag{C.57}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.58}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.59}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(3) \tag{C.60}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(4) \tag{C.61}$$
$$\mathrm{SU}(6) \supset \mathrm{USp}(6) \tag{C.62}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(3) \tag{C.63}$$

of which only the first three contain G_{std}. By inspection, it now follows that for $n \geq 2$, an SU(n) instanton will break too much of the gauge group to preserve G_{std}. Moreover, it follows from lines (C.57)–(C.59) that up to linear combinations of the U(1) charge for the other breaking patterns, it is enough to analyze the $\mathrm{U}(1)^2$ instanton configuration which breaks SU(6) via the nested sequence $\mathrm{SU}(6) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \supset G_{\mathrm{std}} \times \mathrm{U}(1)$. Restricting to $\mathrm{U}(1)^2$ valued instanton configurations, the decomposition of the one two and three index anti-symmetric and adjoint representations of SU(6) are:

$$\mathrm{SU}(6) \supset \mathrm{SU}(5) \times [\mathrm{U}(1)] \supset [\mathrm{SU}(3) \times \mathrm{SU}(2) \times [\mathrm{U}(1)]] \times [\mathrm{U}(1)] \tag{C.64}$$

$$6 \to (1,1)_{0,5} + (3,1)_{-2,-1} + (1,2)_{3,-1} \tag{C.65}$$

$$15 \to (1,2)_{3,-4} + (3,1)_{-2,-4} + (1,1)_{6,2} \tag{C.66}$$

$$+ (\overline{3},1)_{-4,-3} + (3,2)_{1,2} \tag{C.67}$$

$$20 \to (1,1)_{6,-3} + (\overline{3},1)_{-4,-3} + (3,2)_{1,-3} \tag{C.68}$$

$$+ (1,1)_{-6,3} + (3,1)_{4,3} + (\overline{3},2)_{-1,3} \tag{C.69}$$

$$35 \to (1,1)_0 + (1,2)_{3,6} + (3,1)_{-2,6} \tag{C.70}$$

$$+ (1,2)_{-3,-6} + (\overline{3},1)_{2,-6} + (1,1)_{0,0} + (1,3)_{0,0} \tag{C.71}$$

$$+ (3,2)_{-5,0} + (\overline{3},2)_{5,0} + (8,1)_{0,0}. \tag{C.72}$$

The above decomposition illustrates the fact that there are a priori different ways in which the representation content of the MSSM can be packaged into higher dimensional representations.

We now determine all possible choices consistent with obtaining the correct spectrum and interaction terms. We first require that at least one linear combination of the U(1) charges may be identified with $\mathrm{U}(1)_Y$ of the Standard Model. Labeling the U(1) charges as a and b, this implies that the charges of the MSSM fields must satisfy the relations:

$$E : 5b = \pm 6 \text{ or } 6a + 2b = \pm 6 \text{ or } 6a - 3b = \pm 6 \tag{C.73}$$

$$Q : a + 2b = \pm 1 \text{ or } a - 3b = \pm 1 \text{ or } -5a = \pm 1 \tag{C.74}$$

$$U : 2a + b = -4 \text{ or } 2a + 4b = -4 \text{ or } -4a - 3b = -4 \text{ or } 2a - 6b = -4 \tag{C.75}$$

$$D : 2a + b = 2 \text{ or } 2a + 4b = 2 \text{ or } -4a - 3b = 2 \text{ or } 2a - 6b = 2 \tag{C.76}$$

$$H_d, L : 3a - b = \pm 3 \text{ or } 3a + 6b = \pm 3 \tag{C.77}$$

$$H_u : 3a - b = \pm 3 \text{ or } 3a + 6b = \pm 3\,. \tag{C.78}$$

First suppose that the E-relation $5b = \pm 6$ holds. In this case, the remaining candidate solutions for a are:

$$Q \Longrightarrow \pm a = \frac{17}{5} \text{ or } \frac{7}{5} \text{ or } \frac{23}{5} \text{ or } \frac{13}{5} \text{ or } \frac{1}{5} \tag{C.79}$$

$$L \Longrightarrow \pm a = \frac{7}{5} \text{ or } \frac{3}{5} \text{ or } \frac{17}{5} \tag{C.80}$$

$$D \Longrightarrow a = \frac{8}{5} \text{ or } \frac{2}{5} \text{ or } -\frac{7}{5} \text{ or } \frac{17}{5} \text{ or } -\frac{7}{5} \text{ or } \frac{4}{5} \text{ or } \frac{23}{5} \text{ or } -\frac{13}{5} \tag{C.81}$$

$$U \Longrightarrow a = -\frac{13}{5} \text{ or } -\frac{7}{5} \text{ or} -\frac{22}{5} \text{ or } \frac{2}{5} \text{ or } \frac{1}{10} \text{ or } \frac{19}{10} \text{ or } \frac{8}{5} \text{ or} -\frac{28}{5} \tag{C.82}$$

so that the only common solution to all of the above conditions requires $a = -7/5$. Note, however, that this is an inconsistent assignment because whereas the U condition requires $a = -7/5$ and $b = -6/5$, the Q condition requires $a = -7/5$ and $b = +6/5$.

Next suppose that the E-relation $6a+2b=\pm 6$ holds. In this case, b is now determined by the relations:

$$Q \Longrightarrow b = \pm\frac{6}{5} \text{ or } \pm\frac{3}{5} \text{ or } \pm\frac{18}{5} \text{ or } \pm\frac{12}{5} \text{ or } 0 \tag{C.83}$$

$$U \Longrightarrow b = -18 \text{ or } -6 \text{ or } -\frac{9}{5} \text{ or } -\frac{3}{5} \text{ or } \frac{24}{5} \text{ or } \frac{9}{10} \text{ or } \frac{3}{10} \text{ or } 0. \tag{C.84}$$

It thus follows that in this case that the only consistent choice of $\mathrm{U}(1)_Y$ requires $b=0$. Note that in this case the U(1) charge assignments match to those of the SU(5) GUT.

Finally, suppose that the E-relation $6a-3b=\pm 6$ holds. In this case, b is now determined by the possible Q-relations to be:

$$Q \Longrightarrow b = \pm\frac{4}{5} \text{ or } \pm\frac{12}{5} \text{ or } \pm\frac{8}{5} \text{ or } 0 \tag{C.85}$$

$$U \Longrightarrow b = -3 \text{ or } -1 \text{ or } \pm\frac{6}{5} \text{ or } \pm\frac{2}{5} \text{ or } \frac{8}{5} \text{ or } 0 \tag{C.86}$$

$$D \Longrightarrow b = 2 \text{ or } \pm\frac{4}{5} \text{ or } -\frac{6}{5} \text{ or } 0 \tag{C.87}$$

so that the only consistent solution requires $b=0$, as before.

C.2.2 SO(10)

We assume that the matter content organizes into the representations 10, 16, $\overline{16}$ and 45 of SO(10). The maximal subgroups of SO(10) which contain G_{std} are:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5)\times \mathrm{U}(1) \tag{C.88}$$

$$\mathrm{SO}(10) \supset \mathrm{SU}(2)\times \mathrm{SU}(2)\times \mathrm{SU}(4) \tag{C.89}$$

$$\mathrm{SO}(10) \supset \mathrm{SO}(9) \tag{C.90}$$

$$\mathrm{SO}(10) \supset \mathrm{SU}(2)\times \mathrm{SO}(7) \tag{C.91}$$

$$\mathrm{SO}(10) \supset \mathrm{SO}(8)\times \mathrm{U}(1) \tag{C.92}$$

$$\mathrm{SO}(10) \supset \mathrm{USp}(4) \tag{C.93}$$

$$\mathrm{SO}(10) \supset \mathrm{USp}(4)\times \mathrm{USp}(4). \tag{C.94}$$

Of the above maximal subgroups, only the first four contain $\mathrm{SU}(3)\times\mathrm{SU}(2)$ as a subgroup. Whereas lines (C.88) and (C.89) lead to well-known GUTs, the maximal subgroups of lines (C.90) and (C.91) are typically not treated in the GUT literature.

We now demonstrate that no breaking pattern of the latter two cases can yield the MSSM spectrum. In the case $\mathrm{SO}(10)\supset\mathrm{SO}(9)$, the 10, 16, $\overline{16}$ and 45 of SO(10) descend to the 9, 16, and 36 of SO(9). It now follows from the analysis of subsection C.1.3 that no breaking pattern will yield the matter content of the Standard Model.

Next consider the maximal subgroup $\mathrm{SU}(2)\times\mathrm{SO}(7)$. Because there is only one maximal subgroup of SO(7) which contains SU(3), the unique candidate breaking pattern in this case is:

$$\mathrm{SO}(10) \supset \mathrm{SU}(2) \times \mathrm{SO}(7) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \tag{C.95}$$

$$10 \to (3,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.96}$$

$$16 \to (2,1)_3 + (2,3)_{-1} + (2,1)_{-3} + (2,\overline{3})_1 \tag{C.97}$$

$$45 \to (3,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 \,. \tag{C.98}$$

By inspection, we note that all singlets of SU(3) × SU(2) are also neutral under the U(1) factor. We therefore conclude that such a breaking pattern cannot include E-fields.

We now analyze breaking patterns of the two remaining cases of lines (C.88) and (C.89) which are both well-known in the GUT literature. In the present context, we wish to determine whether a non-standard embedding of the fields in an SO(10) representation could also be consistent with the field content of the MSSM.

SO(10) ⊃ SU(5) × U(1). Consider first the maximal subgroup SU(5) × U(1). In this case, the unique nested sequence of maximal subgroups which contains the gauge group G_{std} is:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \times [\mathrm{U}(1)] \supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times [\mathrm{U}(1)_a] \times [\mathrm{U}(1)_b] \tag{C.99}$$

$$10 \to (1,2)_{3,2} + (3,1)_{-2,2} + (1,2)_{-3,-2} + (\overline{3},1)_{2,-2} \tag{C.100}$$

$$16 \to (1,1)_{0,-5} + (1,2)_{-3,3} + (\overline{3},1)_{2,3} + (1,1)_{6,-1} \tag{C.101}$$

$$+ (\overline{3},1)_{-4,-1} + (3,2)_{1,-1} \tag{C.102}$$

$$45 \to (1,1)_0 + (1,1)_{6,4} + (\overline{3},1)_{-4,4} + (3,2)_{1,4} \tag{C.103}$$

$$+ (1,1)_{-6,-4} + (3,1)_{4,-4} + (\overline{3},2)_{-1,-4} + (1,1)_{0,0} \tag{C.104}$$

$$+ (1,3)_{0,0} + (8,1)_{0,0} + (3,2)_{-5,0} + (\overline{3},2)_{5,0}. \tag{C.105}$$

As usual, we require that at least one linear combination of the U(1) charges may be identified with $\mathrm{U}(1)_Y$ of the Standard Model and that all of the necessary interaction terms of the MSSM are present. We begin by classifying all possible combinations of Q-, U- and D-fields which can yield the gauge invariant combination QUH_u:

	Q	U	H_u	(a,b)
1	$(3,2)_{1,-1}$	$(\overline{3},1)_{2,-2}$	$(2,1)_{-3,3}$	*OUT*
2	$(3,2)_{1,-1}$	$(\overline{3},1)_{2,3}$	$(2,1)_{-3,-2}$	$(-1/5,-6/5)$
3	$(3,2)_{1,-1}$	$(\overline{3},1)_{-4,-1}$	$(2,1)_{3,2}$	$(1,0)$
4	$(3,2)_{1,-1}$	$(\overline{3},1)_{-4,4}$	$(2,1)_{3,-3}$	$(1,0)$
5	$(3,2)_{1,4}$	$(\overline{3},1)_{2,-2}$	$(2,1)_{-3,-2}$	$(-7/5,3/5)$
6	$(3,2)_{1,4}$	$(\overline{3},1)_{2,3}$	*OUT*	*OUT*
7	$(3,2)_{1,4}$	$(\overline{3},1)_{-4,-1}$	$(2,1)_{3,-3}$	$(1,0)$
8	$(3,2)_{1,4}$	$(\overline{3},1)_{-4,4}$	*OUT*	*OUT*
9	$(3,2)_{-5,0}$	$(\overline{3},1)_{2,-2}$	$(2,1)_{3,2}$	$(-1/5,9/5)$
10	$(3,2)_{-5,0}$	$(\overline{3},1)_{2,3}$	$(2,1)_{3,-3}$	$(-1/5,-6/5)$
11	$(3,2)_{-5,0}$	$(\overline{3},1)_{-4,-1}$	*OUT*	*OUT*
12	$(3,2)_{-5,0}$	$(\overline{3},1)_{-4,4}$	*OUT*	*OUT*

(C.106)

In the above list, entries in the H_u column listed by "OUT" indicate that of the available representations, no choice yields a gauge invariant quantity in the parent theory. Similarly, an "OUT" entry in the (a,b) column indicates that no consistent solution of $\mathrm{U}(1)_Y$ exists in this case. We next require that a consistent choice of representation for D and H_d to admit the interaction QDH_d also exists amongst the remaining possibilities:

	Q	D	H_d	(a,b)
2	$(3,2)_{1,-1}$	$(\overline{3},1)_{-4,-1}$	$(2,1)_{3,2}$	$(-1/5,-6/5)$
3	$(3,2)_{1,-1}$	$(\overline{3},1)_{2,3}$	$(2,1)_{-3,-2}$	$(1,0)$
4	$(3,2)_{1,-1}$	$(\overline{3},1)_{2,-2}$	$(2,1)_{-3,3}$	$(1,0)$
5	$(3,2)_{1,4}$	OUT	$(2,1)_{3,2}$	$(-7/5,3/5)$
7	$(3,2)_{1,4}$	OUT	$(2,1)_{-3,3}$	$(1,0)$
9	$(3,2)_{-5,0}$	OUT	$(2,1)_{-3,-2}$	$(-1/5,9/5)$
10	$(3,2)_{-5,0}$	OUT	$(2,1)_{-3,3}$	$(-1/5,-6/5)$

. (C.107)

Of the three remaining possibilities, we next require that the interaction term ELH_d be present:

	E	L	H_d	(a,b)
$2a$	$(1,1)_{0,-5}$	$(1,2)_{-3,3}$	$(2,1)_{3,2}$	$(-1/5,-6/5)$
$2b$	$(1,1)_{-6,-4}$	$(1,2)_{3,2}$	$(2,1)_{3,2}$	$(-1/5,-6/5)$
$3a$	$(1,1)_{6,-1}$	$(1,2)_{-3,-2}$	$(2,1)_{-3,-2}$	$(1,0)$
$3b$	$(1,1)_{6,4}$	$(1,2)_{-3,3}$	$(2,1)_{-3,-2}$	$(1,0)$
$4a$	$(1,1)_{6,-1}$	$(1,2)_{-3,-2}$	$(2,1)_{-3,3}$	$(1,0)$
$4b$	$(1,1)_{6,4}$	OUT	$(2,1)_{-3,3}$	$(1,0)$

. (C.108)

We therefore conclude that there are in fact five distinct ways in which the field content of the MSSM can be packaged in representations of SO(10). We note in particular that in some cases, the chiral matter of the MSSM does not descend from either of the spinor representations of SO(10). The above classification can also be obtained without imposing the condition that non-trivial interaction terms be present in the superpotential. Indeed, by listing all possible consistent choices of U(1) charge assignments, we arrive at the same list of admissible configurations. Finally, we note that the choice $b=0$ corresponds to the breaking pattern where $\mathrm{U}(1)_Y$ embeds in SU(5) and the other consistent choice corresponds to the flipped embedding of hypercharge [67].

SO(10) ⊃ SU(2) × SU(2) × SU(4). We next analyze the other nested sequence of maximal subgroups given by decomposing SO(10) as:

$$\mathrm{SO}(10)\supset\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(4)\supset\mathrm{SU}(2)\times\mathrm{SU}(2)\times[\mathrm{SU}(3)\times\mathrm{U}(1)] \tag{C.109}$$
$$10\rightarrow(2,2,1)_0+(1,1,3)_2+(1,1,\overline{3})_{-2} \tag{C.110}$$
$$16\rightarrow(2,1,1)_3+(2,1,3)_{-1}+(2,1,1)_{-3}+(2,1,\overline{3})_1 \tag{C.111}$$
$$45\rightarrow(3,1,1)_0+(1,3,1)_0+(1,1,1)_0+(1,1,3)_{-4} \tag{C.112}$$
$$+(1,1,\overline{3})_4+(1,1,8)_0+(2,2,3)_2+(2,2,\overline{3})_{-2}. \tag{C.113}$$

While an SU(2) instanton configuration can indeed yield the gauge group $G_{\rm std}$, we note that the putative $\mathrm{U}(1)_Y$ would then be incorrect. It thus follows that it is enough to consider $\mathrm{U}(1) \times \mathrm{U}(1)$ instanton configurations. Because the representation content of this decomposition is identical to that of the previous case, we conclude that there are again two possible ways to package the MSSM fields into SO(10) representations.

C.2.3 SO(11)

We assume that the matter content organizes into the representations 11, 32 and 55 of SO(11). The maximal subgroups of SO(11) are:

$$\mathrm{SO}(11) \supset \mathrm{SO}(10) \tag{C.114}$$
$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SO}(8) \tag{C.115}$$
$$\mathrm{SO}(11) \supset \mathrm{USp}(4) \times \mathrm{SU}(4) \tag{C.116}$$
$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(7) \tag{C.117}$$
$$\mathrm{SO}(11) \supset \mathrm{SO}(9) \times \mathrm{U}(1) \tag{C.118}$$
$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \tag{C.119}$$

so that only the first five maximal subgroups contain $G_{\rm std}$.

SO(11) ⊃ SO(10). In the case $\mathrm{SO}(11) \supset \mathrm{SO}(10)$, the representations of SO(11) decompose as:

$$\mathrm{SO}(11) \supset \mathrm{SO}(10) \tag{C.120}$$
$$11 \to 1 + 10 \tag{C.121}$$
$$32 \to 16 + \overline{16} \tag{C.122}$$
$$55 \to 10 + 45 \tag{C.123}$$

so that all of the analysis of breaking patterns performed for SO(10) carries over to this case as well. In this case, it less clear whether the resulting matter spectrum can be chiral, but all matter fields of the MSSM can indeed be present.

SO(11) ⊃ SU(2) × SO(8). In the case $\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SO}(8)$, the representation content of SO(11) decomposes as:

$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SO}(8) \tag{C.124}$$
$$11 \to (3,1) + (1,8^v) \tag{C.125}$$
$$32 \to (2,8^s) + (2,8^c) \tag{C.126}$$
$$55 \to (3,1) + (1,28) + (3,8^v). \tag{C.127}$$

The two maximal subgroups of SO(8) which contain an SU(3) factor are $\mathrm{SU}(4) \times \mathrm{U}(1)$ and $\mathrm{SO}(7) \supset \mathrm{SU}(4)$.

SO(11) ⊃ SU(2) × SO(8) ⊃ SU(2) × [SU(4) × [U(1)]] The decomposition to SU(2) × [SU(4) × [U(1)]] is:

$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SO}(8) \supset \mathrm{SU}(2) \times [\mathrm{SU}(4) \times [\mathrm{U}(1)]] \tag{C.128}$$
$$11 \to (3,1)_0 + (1,1)_2 + (1,1)_{-2} + (1,6)_0 \tag{C.129}$$
$$32 \to (2,8^s) + (2,8^c) \to (2,4)_1 + (2,\overline{4})_{-1} + (2,4)_{-1} \tag{C.130}$$
$$+ (2,\overline{4})_1 \tag{C.131}$$
$$55 \to (3,1)_0 + (1,1)_0 + (1,6)_2 + (1,6)_{-2} + (1,15)_0 \tag{C.132}$$
$$+ (3,1)_2 + (3,1)_{-2} + (3,6)_0 \tag{C.133}$$

so that the decomposition to $G_{\text{std}} \times \mathrm{U}(1)$ along this path is:

$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]_a \times [\mathrm{U}(1)]_b] \tag{C.134}$$
$$11 \to (3,1)_{0,0} + (1,1)_{0,2} + (1,1)_{0,-2} + (1,3)_{2,0} + (1,\overline{3})_{-2,0} \tag{C.135}$$
$$32 \to (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,-1} + (2,\overline{3})_{1,-1} \tag{C.136}$$
$$+ (2,1)_{3,-1} + (2,3)_{-1,-1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.137}$$
$$55 \to (3,1)_{0,0} + (1,1)_{0,0} + (1,3)_{2,2} + (1,\overline{3})_{-2,2} \tag{C.138}$$
$$+ (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} + (1,1)_{0,0} + (1,3)_{-4,0} \tag{C.139}$$
$$+ (1,\overline{3})_{4,0} + (1,8)_{0,0} + (3,1)_{0,2} + (3,1)_{0,-2} \tag{C.140}$$
$$+ (3,3)_{2,0} + (3,\overline{3})_{-2,0}. \tag{C.141}$$

In order to achieve the correct $\mathrm{U}(1)_Y$ charge assignment for the E-fields and Q-fields, we require:

$$2b = \pm 6 \tag{C.142}$$
$$-a \pm b = 1 \tag{C.143}$$

so that:

$$b = \pm 3 \tag{C.144}$$
$$a = -4 \text{ or } 2. \tag{C.145}$$

In order to achieve the correct $\mathrm{U}(1)_Y$ charge assignment for the L-fields, we must also require:

$$\pm 3a \pm b = \pm 3 \tag{C.146}$$

so that $a = 2$ and without loss of generality, we may choose a sign convention for b so that $b = 3$. In this case, the candidate representations for Q, D and H_d are:

Q	D	H_d
$(2,3)_{-1,1}$	$(1,\overline{3})_{-2,2}$	$(2,1)_{-3,-1}$

(C.147)

so that the product QDH_d is not neutral under $\mathrm{U}(1)_a$. We therefore conclude that this breaking pattern cannot yield the spectrum of the Standard Model.

SO(11) ⊃ SU(2) × SO(7) ⊃ SU(2) × SU(4) ⊃ SU(2) × SU(3) × U(1) In this case, breaking to G_{std} proceeds via the nested sequence:

$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{SO}(7) \tag{C.148}$$
$$\supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.149}$$
$$11 \to (3,1)_0 + (1,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.150}$$
$$32 \to (2,1)_3 + (2,3)_{-1} + (2,1)_{-3} + (2,\overline{3})_1 + (2,1)_3 \tag{C.151}$$
$$+ (2,3)_{-1} + (2,1)_{-3} + (2,\overline{3})_1 \tag{C.152}$$
$$55 \to (3,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} + (1,3)_2 \tag{C.153}$$
$$+ (1,\overline{3})_{-2} + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 \tag{C.154}$$
$$+ (3,1)_0 + (3,1)_0 + (3,3)_2 + (3,\overline{3})_{-2} \tag{C.155}$$

By inspection, the above decomposition does not contain any E-fields. We therefore conclude that in all cases, breaking patterns of SO(11) with maximal subgroup SU(2) × SO(8) cannot contain G_{std}.

SO(11) ⊃ USp(4) × SU(4). Because USp(4) does not contain SU(3) as a subgroup, it follows that in this case, SU(4) must decompose to SU(3) × U(1). The decomposition must therefore proceed via the path:

$$\mathrm{SO}(11) \supset \mathrm{USp}(4) \times \mathrm{SU}(4) \supset \mathrm{USp}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.156}$$
$$11 \to (5,1) + (1,6) \to (5,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.157}$$
$$32 \to (4,4) + (4,\overline{4}) \to (4,1)_3 + (4,3)_{-1} + (4,1)_{-3} \tag{C.158}$$
$$55 \to (10,1) + (1,15) + (5,6) \to (10,1)_0 + (1,1)_0 \tag{C.159}$$
$$+ (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0. \tag{C.160}$$

To proceed further, we specify a maximal subgroup of USp(4) among the ones listed in lines (C.21)–(C.23). Because a given instanton configuration must preserve the non-abelian factor SU(3) × SU(2) of the G_{std}, we conclude that only the first two are viable breaking patterns.

SO(11) ⊃ USp(4) × SU(4) ⊃ USp(4) × SU(3) × U(1) ⊃ [SU(2) × SU(2)] × [SU(3) × U(1)] In this case, the decomposition of the matter content contains the representation content of the breaking pattern SO(10) ⊃ SU(2) × SU(2) × SU(4). Explicitly:

$$\mathrm{SO}(11) \supset \mathrm{USp}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(2) \times \mathrm{SU}(2)] \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \tag{C.161}$$
$$11 \to (1,1,1)_0 + (2,2,1)_0 + (1,1,3)_2 + (1,1,\overline{3})_{-2} \tag{C.162}$$
$$32 \to (2,1,1)_3 + (2,1,3)_{-1} + (1,2,1)_3 + (2,1,3)_{-1} \tag{C.163}$$
$$+ (2,1,1)_{-3} + (2,1,\overline{3})_1 + (1,2,1)_{-3} + (2,1,\overline{3})_1 \tag{C.164}$$
$$55 \to (3,1,1)_0 + (1,1,1)_0 + (1,1,3)_{-4} + (1,1,\overline{3})_4 \tag{C.165}$$
$$+ (1,1,8)_0 + (1,1,3)_2 + (1,1,\overline{3})_{-2} + (2,2,3)_2 \tag{C.166}$$
$$+ (2,2,\overline{3})_{-2} + (1,3,1)_0 + (2,2,1)_0. \tag{C.167}$$

It follows that the analysis of breaking patterns for SO(10) directly carries over to this case as well.

SO(11) ⊃ USp(4) × SU(3) × U(1) ⊃ [SU(2) × U(1)] × [SU(3) × U(1)]

While this is seemingly quite similar to the breaking pattern described previously, we now show that the embedding of the U(1) factor in USp(4) does not admit an embedding of the matter content of the Standard Model. To this end, we first decompose SO(11) via:

$$\mathrm{SO}(11) \supset \mathrm{USp}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(2) \times [\mathrm{U}(1)]_a] \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]_b] \tag{C.168}$$
$$11 \to (1,1)_{2,0} + (1,1)_{-2,0} + (3,1)_{0,0} + (1,3)_{0,2} + (1,\overline{3})_{0,-2} \tag{C.169}$$
$$32 \to (2,1)_{1,3} + (2,3)_{1,-1} + (2,1)_{-1,3} + (2,3)_{-1,-1} + (2,1)_{-1,-3} \tag{C.170}$$
$$+ (2,\overline{3})_{-1,1} + (2,1)_{1,-3} + (2,\overline{3})_{1,1} \tag{C.171}$$
$$55 \to (1,1)_{0,0} + (3,1)_{0,0} + (3,1)_{2,0} + (3,1)_{-2,0} + (1,1)_{0,0} \tag{C.172}$$
$$+ (1,3)_{0,-4} + (1,\overline{3})_{0,4} + (1,8)_{0,0} + (1,3)_{2,2} + (1,\overline{3})_{2,-2} \tag{C.173}$$
$$+ (1,\overline{3})_{-2,-2} + (1,3)_{-2,2} + (3,3)_{0,2} + (3,\overline{3})_{0,-2}. \tag{C.174}$$

It follows from the above decomposition that the E-fields correspond to the representation $(1,1)_{\pm 2,0}$ of the above decomposition. It thus follows that $a = \pm 3$. Because the Q-fields correspond to the representation $(2,3)_{\pm 1,-1}$ and the L fields correspond to the representation $(2,1)_{\pm 1,\pm 3}$, we conclude that without loss of generality, fixing the sign of a to be positive so that $a = +3$, there is a unique linear combination of U(1) charges so that $a = 3$ and $b = 2$. The field content of the MSSM thus descends from the above representations as:

E	Q	U	D	L	H_u	H_d
$(1,1)_{2,0}$	$(2,3)_{1,-1}$	$(1,\overline{3})_{0,-2}$	$(1,\overline{3})_{2,-2}$	$(2,1)_{1,-3}$	$(2,1)_{-1,3}$	$(2,1)_{1,-3}$

. (C.175)

By inspection, we note that whereas the product QUH_u is indeed invariant under all gauge group factors, QDH_d violates $\mathrm{U}(1)_b$. We therefore conclude that the above breaking pattern cannot yield the MSSM.

SO(11) ⊃ SU(2) × SU(2) × SO(7). Because there is a single maximal subgroup of SO(7) which contains SU(3), we find that the unique breaking pattern which can reproduce G_{std} proceeds as:

$$\mathrm{SO}(11) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(7) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \tag{C.176}$$
$$11 \to (2,2,1) + (1,1,1) + (1,1,6) \tag{C.177}$$
$$32 \to (1,2,4) + (1,2,\overline{4}) + (2,1,4) + (2,1,\overline{4}) \tag{C.178}$$
$$55 \to (3,1,1) + (1,3,1) + (2,2,1) + (2,2,6) + (1,1,6) + (1,1,15)\,. \tag{C.179}$$

By inspection, this decomposition again contains all of the matter content of the SO(10) breaking pattern which proceeds via SO(10) ⊃ SU(2) × SU(2) × SU(4). We therefore conclude that the analysis of the breaking patterns via instantons is identical to this case.

SO(11) ⊃ SO(9) × U(1). The final maximal subgroup which contains G_{std} is given by SO(9) × U(1). In this case, SU(2) × SU(4) is the only maximal subgroup of SO(9) which contains the product SU(3) × SU(2). Decomposing with respect to this path yields:

$$\mathrm{SO}(11) \supset \mathrm{SO}(9) \times [\mathrm{U}(1)]_b \supset [\mathrm{SU}(2) \times \mathrm{SU}(4)] \times [\mathrm{U}(1)]_b \tag{C.180}$$
$$\supset [\mathrm{SU}(2) \times \mathrm{SU}(3) \times [\mathrm{U}(1)]_a] \times [\mathrm{U}(1)]_b \tag{C.181}$$
$$11 \to (1,1)_{0,-2} + (1,1)_{0,2} + (3,1)_{0,0} + (1,3)_{2,0} + (1,\overline{3})_{-2,0} \tag{C.182}$$
$$32 \to (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.183}$$
$$+ (2,1)_{-3,-1} + (2,\overline{3})_{1,-1} + (2,1)_{3,-1} + (2,3)_{-1,-1} \tag{C.184}$$
$$55 \to (1,1)_{0,0} + (3,1)_{0,2} + (1,3)_{2,2} + (1,\overline{3})_{-2,2} + (3,1)_{0,-2} \tag{C.185}$$
$$+ (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} + (3,1)_{0,0} + (3,3)_{2,0} + (3,\overline{3})_{-2,0} \tag{C.186}$$
$$+ (1,1)_{0,0} + (1,3)_{-4,0} + (1,\overline{3})_{4,0} + (1,8)_{0,0}. \tag{C.187}$$

In this case, the E-fields must correspond to the representation $(1,1)_{0,\pm 2}$. This implies the relation $b = \pm 3$. Moreover, because the Q and L-fields respectively correspond to the representations $(2,3)_{-1,\pm 1}$ and $(2,1)_{\pm 3,\pm 1}$, it follows that without loss of generality $a = 2$ and $b = +3$ is the unique choice of U(1) charges which can yield the correct value of $\mathrm{U}(1)_Y$ for all fields. In this case, the representation content of the Q, D and H_d fields is uniquely determined to be:

Q	D	H_d
$(2,3)_{-1,1}$	$(1,\overline{3})_{-2,2}$	$(2,1)_{-3,-1}$

. (C.188)

Because the product QDH_d violates $\mathrm{U}(1)_b$, we conclude that the corresponding breaking pattern cannot lead to the MSSM.

C.3 Rank six

We now proceed to the classification of all breaking patterns of rank six groups. Because it is the case of primary phenomenological interest in many cases, we begin our analysis with breaking patterns of E_6. We next determine all possible breaking patterns of SU(7) and conclude with an analysis of breaking patterns of SO(12).

C.3.1 E_6

The non-trivial representations of E_6 which can descend from the adjoint representation of E_8 are the 27, $\overline{27}$ and 78 of E_6. The maximal subgroups of E_6 are:

$$E_6 \supset \mathrm{SO}(10) \times \mathrm{U}(1) \tag{C.189}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \tag{C.190}$$
$$E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \tag{C.191}$$
$$E_6 \supset \mathrm{USp}(8) \tag{C.192}$$
$$E_6 \supset F_4 \tag{C.193}$$
$$E_6 \supset \mathrm{SU}(3) \times G_2 \tag{C.194}$$
$$E_6 \supset G_2 \tag{C.195}$$
$$E_6 \supset \mathrm{SU}(3). \tag{C.196}$$

Of the above configurations, only the maximal subgroups of lines (C.189)–(C.194) contain G_{std}. In particular, the first three breaking patterns can descend to more conventional GUT theories. We begin our analysis by demonstrating that none of the remaining possibilities can produce a consistent embedding of the MSSM.

$E_6 \supset$ USp(8). The maximal subgroups of USp(8) are:

$$\mathrm{USp}(8) \supset \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.197}$$
$$\mathrm{USp}(8) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \tag{C.198}$$
$$\mathrm{USp}(8) \supset \mathrm{USp}(4) \times \mathrm{USp}(4) \tag{C.199}$$
$$\mathrm{USp}(8) \supset \mathrm{SU}(2) \tag{C.200}$$
$$\mathrm{USp}(8) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{C.201}$$

Of these possibilities, only line (C.198) contains SU(3) × SU(2). Further, by inspection of lines (C.17)–(C.20), the only maximal subgroup of USp(6) which contains SU(3) is:

$$\mathrm{USp}(6) \supset \mathrm{SU}(3) \times \mathrm{U}(1). \tag{C.202}$$

In this case, the unique candidate breaking pattern is:

$$E_6 \supset \mathrm{USp}(8) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \tag{C.203}$$

which is obtained by a non-trivial U(1) instanton in the USp(6) factor. In this case, the representations of E_6 decompose as:

$$E_6 \supset \mathrm{USp}(8) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \tag{C.204}$$
$$27 \to (2,3)_1 + (2,\overline{3})_{-1} + (1,3)_{-2} + (1,\overline{3})_2 + (1,8)_0 + (1,1)_0 \tag{C.205}$$
$$78 \to (3,1)_0 + (1,1)_0 + (1,6)_2 + (1,\overline{6})_{-2} + (1,8)_0 + (2,3)_1 \tag{C.206}$$
$$+ (2,\overline{3})_{-1} + (1,3)_{-2} + (1,\overline{3})_2 + (1,8)_0 + (2,1)_3 + (2,1)_{-3} \tag{C.207}$$
$$+ (2,6)_{-1} + (2,\overline{6})_1. \tag{C.208}$$

By inspection, all singlets of SU(3) × SU(2) are neutral under the only U(1) factor so that the resulting model cannot contain any E-fields.

$E_6 \supset F_4$. The representation content of E_6 decomposes under F_4 as:

$$E_6 \supset F_4 \tag{C.209}$$
$$27 \to 26 + 1 \tag{C.210}$$
$$78 \to 26 + 52. \tag{C.211}$$

Returning to our previous analysis of breaking patterns for F_4, we therefore conclude that this breaking pattern cannot produce the correct matter content of the MSSM.

$E_6 \supset$ SU(3) $\times$ G_2. Although G_2 contains SU(3) as a maximal subgroup, it is not possible to arrange for an instanton configuration to break G_2 to SU(3). For this reason, we conclude that the SU(3) factor of $G_{\rm std}$ must be identified with the SU(3) factor of the maximal subgroup SU(3) $\times$ G_2 of E_6. In this case, it now follows that the factor SU(2) $\times$ U(1) must descend from G_2. Returning to lines (C.24)–(C.26), it follows that the maximal subgroups SU(3) and SU(2) $\times$ SU(2) contain SU(2) $\times$ U(1).

First consider the decomposition of representations of E_6 via the nested sequence of maximal subgroups:

$$E_6 \supset \mathrm{SU}(3) \times G_2 \supset \mathrm{SU}(3) \times [\mathrm{SU}(3)] \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times [\mathrm{U}(1)]] \tag{C.212}$$

$$27 \to (\overline{6},1)_0 + (3,1)_0 + (3,2)_1 + (3,1)_{-2} + (3,2)_{-1} + (3,1)_2 \tag{C.213}$$

$$78 \to (8,1)_0 + (1,1)_{-2} + (1,2)_1 + (1,1)_2 + (1,2)_{-1} \tag{C.214}$$

$$+ (1,1)_0 + (1,2)_3 + (1,2)_{-3} + (1,3)_0. \tag{C.215}$$

Because the ratio of the U(1) charge for the candidate E- and Q-fields does not equal six, we conclude that this is not a viable breaking pattern.

Next consider the decomposition associated with the nested sequence of maximal subgroups:

$$E_6 \supset \mathrm{SU}(3) \times G_2 \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{SU}(2)] \tag{C.216}$$

$$27 \to (\overline{6},1,1) + (3,1,3) + (3,2,2) \tag{C.217}$$

$$78 \to (8,1,1) + (1,1,3) + (1,3,1) + (1,2,4) + (8,1,3) + (8,2,2). \tag{C.218}$$

Decomposing the above representations with respect to a U(1) subgroup of either SU(2) factor, we find that the ratio of U(1) charges for the candidate E- and Q-fields again does not equal six. Hence, neither nested sequence of maximal subgroups yields the correct spectrum of the MSSM.

$E_6 \supset$ SU(3) $\times$ SU(3) $\times$ SU(3). In order to make the $\mathbb{Z}_3$ outer automorphism of E_6 more manifest, we assume that the decomposition of E_6 to the maximal subgroup SU(3) $\times$ SU(3) $\times$ SU(3) is given by:

$$E_6 \supset \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3 \tag{C.219}$$

$$27 \to (\overline{3},3,1) + (3,1,\overline{3}) + (1,\overline{3},3) \tag{C.220}$$

$$\overline{27} \to (3,\overline{3},1) + (\overline{3},1,3) + (1,3,\overline{3}) \tag{C.221}$$

$$78 \to (8,1,1) + (1,8,1) + (1,1,8) + (3,3,3) + (\overline{3},\overline{3},\overline{3}). \tag{C.222}$$

While it is also common to conjugate the representation content of the third SU(3) factor, this is a choice of convention. Indeed, because of the $\mathbb{Z}_3$ outer automorphism, without loss of generality we require that the first SU(3) factor is common to $G_{\rm std}$ as well. First note that while an SU(3) $\times$ U(1) instanton can break E_6 to $G_{\rm std}$, we note that the resulting U(1) factor of $G_{\rm std}$ must descend from one of the remaining SU(3) factors. By inspection of the above decomposition of line (C.219), the purported U(1)$_Y$ is incorrect.

To proceed further, we next consider the maximal subgroups of the last two SU(3) factors. The maximal subgroups of SU(3) are:

$$a) : \mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.223}$$

$$b) : \mathrm{SU}(3) \supset \mathrm{SU}(2). \tag{C.224}$$

We therefore conclude that there are four distinct maximal subgroups of $\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ which can potentially yield G_{std}. Moreover, in order to achieve the subgroup $\mathrm{SU}(2) \times \mathrm{U}(1)$ of G_{std}, we must assume that at least one SU(3) factor descends to a maximal subgroup via line (C.223).

$E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \supset \mathrm{SU}(3) \times [\mathrm{SU}(2)] \times [\mathrm{SU}(2) \times \mathrm{U}(1)]$

We first treat the nested sequence of maximal subgroups where the second SU(3) factor descends to SU(2) as in line (C.224) while the third descends to $\mathrm{SU}(2) \times \mathrm{U}(1)$ as in line (C.223). Because interchanging the last two SU(3) factors of $E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ complex conjugates all representations, a similar analysis will hold in that case as well. The representation content of E_6 decomposes as:

$$E_6 \supset \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3 \supset \mathrm{SU}(3)_1 \times [\mathrm{SU}(2)]_2 \times [\mathrm{SU}(2) \times \mathrm{U}(1)]_3 \tag{C.225}$$

$$27 \to (\overline{3}, 3, 1) + (3, 1, 1_2) + (3, 1, 2_{-1}) + (1, 3, 1_{-2}) + (1, 3, 2_1) \tag{C.226}$$

$$78 \to (8, 1, 1) + (1, 3, 1) + (1, 5, 1) + (1, 1, 1_0) + (1, 1, 2_3) + (1, 1, 2_{-3}) \tag{C.227}$$

$$+ (1, 1, 3_0) + (3, 3, 1_2) + (3, 3, 2_{-1}) + (\overline{3}, 3, 1_{-2}) + (\overline{3}, 3, 2_1). \tag{C.228}$$

There are several ways in which an instanton configuration can yield the gauge group G_{std}. First consider configurations obtained via a non-trivial SU(2) instanton configuration. Because the SU(2) factor of $\mathrm{SU}(3)_2$ either breaks completely or to a U(1) subgroup of SU(2), we conclude that only SU(2) instantons with values in the factor $\mathrm{SU}(3)_2$ of line (C.219) can preserve the gauge group G_{std}. In this case, the U(1) charge assignments for the Q- and E-fields are incompatible with the $\mathrm{U}(1)_Y$ assignments of the Standard Model.

Next consider abelian instanton configurations which break one of the SU(2) factors. Decomposing the factor $\mathrm{SU}(2)_2$ with respect to a U(1) subgroup, the resulting representation content is:

$$E_6 \supset \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3 \supset \mathrm{SU}(3)_1 \times [\mathrm{SU}(2)]_2 \times [\mathrm{SU}(2) \times \mathrm{U}(1)]_3 \tag{C.229}$$

$$\supset \mathrm{SU}(3)_1 \times [\mathrm{U}(1)_a]_2 \times [\mathrm{SU}(2) \times \mathrm{U}(1)_b]_3 \tag{C.230}$$

$$27 \to (\overline{3}, 1_2, 1_0) + (\overline{3}, 1_{-2}, 1_0) + (\overline{3}, 1_0, 1_0) + (3, 1_0, 1_2) + (3, 1_0, 2_{-1}) \tag{C.231}$$

$$+ (1, 1_2, 1_{-2}) + (1, 1_{-2}, 1_{-2}) + (1, 1_0, 1_{-2}) + (1, 1_2, 2_1) + (1, 1_{-2}, 2_1) \tag{C.232}$$

$$+ (1, 1_0, 2_1) \tag{C.233}$$

$$78 \to (8, 1_0, 1_0) + (1, 1_2, 1_0) + (1, 1_{-2}, 1_0) + (1, 1_0, 1_0) + (1, 1_4, 1_0) \tag{C.234}$$

$$+ (1, 1_2, 1_0) + (1, 1_0, 1_0) + (1, 1_{-2}, 1_0) + (1, 1_{-4}, 1_0) + (1, 1_0, 1_0) \tag{C.235}$$

$$+ (1, 1_0, 2_3) + (1, 1_0, 2_{-3}) + (1, 1_0, 3_0) + (3, 1_2, 1_2) + (3, 1_{-2}, 1_2) \tag{C.236}$$

$$+ (3, 1_0, 1_2) + (3, 1_2, 2_{-1}) + (3, 1_{-2}, 2_{-1}) + (3, 1_0, 2_{-1}) + (\overline{3}, 1_{-2}, 1_{-2}) \tag{C.237}$$

$$+ (\overline{3}, 1_2, 1_{-2}) + (\overline{3}, 1_0, 1_{-2}) + (\overline{3}, 1_{-2}, 2_1) + (\overline{3}, 1_2, 2_1) + (\overline{3}, 1_0, 2_1). \tag{C.238}$$

The representation content of each MSSM field therefore descends from the following representations:

$$E : (1, 1_{\pm 2}, 1_{\pm 2}) \tag{C.239}$$

$$Q : (3, 1_0, 2_{-1}) \text{ or } (3, 1_{\pm 2}, 2_{-1}) \tag{C.240}$$

$$H_d, L : (1, 1_{\pm 2}, 2_{\pm 1}) \text{ or } (1, 1_0, 2_{\pm 3}) \tag{C.241}$$

$$U : (\overline{3}, 1_{\pm 2}, 1_0) \text{ or } (\overline{3}, 1_0, 1_{-2}) \text{ or } (\overline{3}, 1_{\pm 2}, 1_{-2}) \tag{C.242}$$

$$D : (\overline{3}, 1_{\pm 2}, 1_0) \text{ or } (\overline{3}, 1_0, 1_{-2}) \text{ or } (\overline{3}, 1_{\pm 2}, 1_{-2}) \tag{C.243}$$

$$H_u : (1, 1_{\pm 2}, 2_{\pm 1}) \text{ or } (1, 1_0, 2_{\pm 3}). \tag{C.244}$$

There are four possible assignments for the Q, U, H_u fields which can yield a non-trivial QUH_u term:

Q	U	H_u
$(3, 1_0, 2_{-1})$	$(\overline{3}, 1_{\pm 2}, 1_0)$	$(1, 1_{\mp 2}, 2_{+1})$
$(3, 1_0, 2_{-1})$	$(\overline{3}, 1_0, 1_{-2})$	$(1, 1_0, 2_{+3})$
$(3, 1_{\pm 2}, 2_{-1})$	$(\overline{3}, 1_{\mp 2}, 1_{-2})$	$(1, 1_0, 2_{+3})$

(C.245)

so that in the first three cases, the $\mathrm{U}(1)_Y$ charge of Q requires $b = -1$ while in the final case the $\mathrm{U}(1)_Y$ charge of H_u requires $b = +1$. In particular, this implies that the second choice of charge assignments in line (C.245) is inconsistent. Next consider the first choice of charge assignments. In order to obtain the correct $\mathrm{U}(1)_Y$ charge assignment for the U-field, we must therefore require $a = \mp 2$. Finally, the fourth choice of charge assignments requires $a = \pm 1$. Of these possible charge assignments, only the first yields a choice consistent with the $\mathrm{U}(1)_Y$ charge of the E-field in line (C.239). We therefore find that $a = -2$ and $b = -1$ where without loss of generality we have chosen a sign for a. It now follows that the only candidate charge assignments for the fields are:

E_{27}	$Q_{27,78}$	U_{27}	$D_{\overline{27},78}$	$L_{\overline{27}}$	H_{u27}	H_{d78}
$(1, 1_{-2}, 1_{-2})$	$(3, 1_0, 2_{-1})$	$(\overline{3}, 1_2, 1_0)$	$(\overline{3}, 1_0, 1_{-2})$	$(1, 1_2, 2_{-1})$	$(1, 1_{-2}, 2_{+1})$	$(1, 1_0, 2_3)$

(C.246)

where we have also indicated the E_6 representation content. The interaction term QUH_u therefore descends from a 27^3 term so that in particular, Q descends from the 27 of E_6. In order to obtain a non-trivial QDH_d term, this in turn requires D to descend from the 78 of E_6 so that we finally obtain the representation content:

E_{27}	Q_{27}	U_{27}	$D_{\overline{27}}$	$L_{\overline{27}}$	H_{u27}	H_{d78}
$(1, 1_{-2}, 1_{-2})$	$(3, 1_0, 2_{-1})$	$(\overline{3}, 1_2, 1_0)$	$(\overline{3}, 1_0, 1_{-2})$	$(1, 1_2, 2_{-1})$	$(1, 1_{-2}, 2_{+1})$	$(1, 1_0, 2_3)$

(C.247)

we therefore conclude that a $\mathrm{U}(1)_2 \times \mathrm{U}(1)_3$ of the above type can indeed yield a spectrum consistent with the MSSM.

$E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{U}(1)] \times [\mathrm{SU}(2) \times \mathrm{U}(1)]$

We next treat the nested sequence of maximal subgroups where the second and third $\mathrm{SU}(3)$ factors of the decomposition $E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ descend to $\mathrm{SU}(2) \times \mathrm{U}(1)$

as in line (C.223). Under this decomposition, the resulting representation content is:

$$
\begin{aligned}
E_6 &= \mathrm{SU}(3)_1 \times \mathrm{SU}(3)_2 \times \mathrm{SU}(3)_3 && \text{(C.248)}\\
&\supset \mathrm{SU}(3)_1 \times [\mathrm{SU}(2) \times \mathrm{U}(1)_a]_2 \times [\mathrm{SU}(2) \times \mathrm{U}(1)_b]_3 && \text{(C.249)}\\
27 &\to (\overline{3}, 1_{-2}, 1_0) + (\overline{3}, 2_1, 1_0) + (3, 1_0, 1_2) + (3, 1_0, 2_{-1}) + (1, 1_2, 1_{-2}) && \text{(C.250)}\\
&+ (1, 2_{-1}, 1_{-2}) + (1, 1_2, 2_1) + (1, 2_{-1}, 2_1) && \text{(C.251)}\\
78 &\to (8, 1_0, 1_0) + (1, 1_0, 1_0) + (1, 2_3, 1_0) + (1, 2_{-3}, 1_0) + (1, 3_0, 1_0) && \text{(C.252)}\\
&+ (1, 1_0, 1_0) + (1, 1_0, 2_3) + (1, 1_0, 2_{-3}) + (1, 1_0, 3_0) + (3, 1_{-2}, 1_{-2}) && \text{(C.253)}\\
&+ (3, 1_{-2}, 2_1) + (3, 2_1, 1_{-2}) + (3, 2_1, 2_1) + (\overline{3}, 1_2, 1_2) + (\overline{3}, 1_2, 2_{-1}) && \text{(C.254)}\\
&+ (\overline{3}, 2_{-1}, 1_2) + (\overline{3}, 2_{-1}, 2_{-1}). && \text{(C.255)}
\end{aligned}
$$

As opposed to previous examples, we now show that a non-abelian instanton can indeed yield the spectrum of the MSSM. To this end, we first show that the representation content under the subgroup $\mathrm{SU}(3)_1 \times [\mathrm{U}(1)_a]_2 \times [\mathrm{SU}(2) \times \mathrm{U}(1)_b]_3$ can yield the desired spectrum. We note that this will then establish the same result for a U(1) instanton which breaks this SU(2) factor to U(1).

The representation content of the candidate fields is given by ignoring the first SU(2) factor:

$$
\begin{aligned}
E &: (1, 1_{2\varepsilon}, 1_{-2\varepsilon}) \text{ or } (1, 2_\varepsilon, 1_{2\varepsilon}) \text{ or } (1, 2_{3\varepsilon}, 1_0) && \text{(C.256)}\\
Q &: (3, 1_0, 2_{-1}) \text{ or } (3, 1_{-2}, 2_1) \text{ or } (3, 2_1, 2_1) && \text{(C.257)}\\
U &: (\overline{3}, 1_{-2}, 1_0) \text{ or } (\overline{3}, 2_1, 1_0) \text{ or } (\overline{3}, 1_0, 1_{-2}) && \text{(C.258)}\\
&\quad \text{or } (\overline{3}, 1_2, 1_2) \text{ or } (\overline{3}, 2_{-1}, 1_2) && \text{(C.259)}\\
D &: (\overline{3}, 1_{-2}, 1_0) \text{ or } (\overline{3}, 2_1, 1_0) \text{ or } (\overline{3}, 1_0, 1_{-2}) && \text{(C.260)}\\
&\quad \text{or } (\overline{3}, 1_2, 1_2) \text{ or } (\overline{3}, 2_{-1}, 1_2) && \text{(C.261)}\\
H_d, H_u, L &: (1, 1_{2\varepsilon}, 2_\varepsilon) \text{ or } (1, 2_{-\varepsilon}, 2_\varepsilon) \text{ or } (1, 1_0, 2_{3\varepsilon}) && \text{(C.262)}
\end{aligned}
$$

where $\varepsilon = \pm 1$. We begin by listing all possible distinct combinations of fields which can potentially descend to the MSSM interaction term QUH_u:

	Q	U	H_u	(a,b)
1	$(3,1_0,2_{-1})$	$(\overline{3},1_{-2},1_0)$	$(1,1_2,2_1)$	$(2,-1)$
2	$(3,1_0,2_{-1})$	$(\overline{3},2_1,1_0)$	$(1,2_{-1},2_1)$	$(-4,-1)$
3	$(3,1_0,2_{-1})$	$(\overline{3},1_0,1_{-2})$	$(1,1_0,2_3)$	OUT
4	$(3,1_0,2_{-1})$	$(\overline{3},1_2,1_2)$	$(1,1_{-2},2_{-1})$	$(-1,-1)$
5	$(3,1_0,2_{-1})$	$(\overline{3},2_{-1},1_2)$	$(1,2_1,2_{-1})$	$(2,-1)$
6	$(3,1_{-2},2_1)$	$(\overline{3},1_{-2},1_0)$	OUT	OUT
7	$(3,1_{-2},2_1)$	$(\overline{3},2_1,1_0)$	$(1,2_1,2_{-1})$	$(-4,-7)$
8	$(3,1_{-2},2_1)$	$(\overline{3},1_0,1_{-2})$	$(1,1_2,2_1)$	OUT
9	$(3,1_{-2},2_1)$	$(\overline{3},1_2,1_2)$	$(1,1_0,2_{-3})$	$(-1,-1)$
10	$(3,1_{-2},2_1)$	$(\overline{3},2_{-1},1_2)$	OUT	OUT
11	$(3,2_1,2_1)$	$(\overline{3},1_{-2},1_0)$	$(1,2_1,2_{-1})$	$(2,-1)$
12	$(3,2_1,2_1)$	$(\overline{3},2_1,1_0)$	$(1,1_{-2},2_{-1})$	$(-4,5)$
13	$(3,2_1,2_1)$	$(\overline{3},1_0,1_{-2})$	$(1,2_{-1},2_1)$	$(-1,2)$
14	$(3,2_1,2_1)$	$(\overline{3},1_2,1_2)$	OUT	OUT
15	$(3,2_1,2_1)$	$(\overline{3},2_{-1},1_2)$	$(1,1_0,2_{-3})$	$(2,-1)$

(C.263)

where we have also solved for the linear combination of $\mathrm{U}(1)_a$ and $\mathrm{U}(1)_b$ consistent with $\mathrm{U}(1)_Y$ charge assignments in the MSSM. Next, we list all possible combinations of fields consistent with the above classification which also allow the interaction term QDH_d.

	Q	U	D
1	$(3,1_0,2_{-1})$	$(\overline{3},1_{-2},1_0)$	$(\overline{3},2_1,1_0)$ or $(\overline{3},1_0,1_{-2})$ or $(\overline{3},1_2,1_2)$
2	$(3,1_0,2_{-1})$	$(\overline{3},2_1,1_0)$	$(\overline{3},1_0,1_{-2})$ or $(\overline{3},2_{-1},1_2)$
4	$(3,1_0,2_{-1})$	$(\overline{3},1_2,1_2)$	$(\overline{3},1_{-2},1_0)$ or $(\overline{3},1_0,1_{-2})$
5	$(3,1_0,2_{-1})$	$(\overline{3},2_{-1},1_2)$	$(\overline{3},2_1,1_0)$ or $(\overline{3},1_0,1_{-2})$ or $(\overline{3},1_2,1_2)$
7	$(3,1_{-2},2_1)$	$(\overline{3},2_1,1_0)$	OUT
9	$(3,1_{-2},2_1)$	$(\overline{3},1_2,1_2)$	$(\overline{3},1_0,1_{-2})$
11	$(3,2_1,2_1)$	$(\overline{3},1_{-2},1_0)$	$(\overline{3},2_1,1_0)$ or $(\overline{3},1_0,1_{-2})$
12	$(3,2_1,2_1)$	$(\overline{3},2_1,1_0)$	$(\overline{3},1_2,1_2)$
13	$(3,2_1,2_1)$	$(\overline{3},1_0,1_{-2})$	$(\overline{3},1_{-2},1_0)$
15	$(3,2_1,2_1)$	$(\overline{3},2_{-1},1_2)$	$(\overline{3},2_1,1_0)$ or $(\overline{3},1_0,1_{-2})$

(C.264)

	H_u	H_d	(a,b)
1	$(1,1_2,2_1)$	$(1,2_{-1},2_1)$ or $(1,1_0,2_3)$ or $(1,1_{-2},2_{-1})$	$(2,-1)$
2	$(1,2_{-1},2_1)$	$(1,1_0,2_3)$ or $(1,2_1,2_{-1})$	$(-4,-1)$
4	$(1,1_{-2},2_{-1})$	$(1,1_2,2_1)$ or $(1,1_0,2_3)$	$(-1,-1)$
5	$(1,2_1,2_{-1})$	$(1,2_{-1},2_1)$ or $(1,1_0,2_3)$ or $(1,1_{-2},2_{-1})$	$(2,-1)$
7	$(1,2_1,2_{-1})$	*OUT*	$(-4,-7)$
9	$(1,1_0,2_{-3})$	$(1,1_2,2_1)$	$(-1,-1)$
11	$(1,2_1,2_{-1})$	$(1,1_{-2},2_{-1})$ or $(1,2_{-1},2_1)$	$(2,-1)$
12	$(1,1_{-2},2_{-1})$	*OUT*	$(-4,5)$
13	$(1,2_{-1},2_1)$	$(1,2_1,2_{-1})$	$(-1,2)$
15	$(1,1_0,2_{-3})$	$(1,1_{-2},2_{-1})$ or $(1,2_{-1},2_1)$	$(2,-1)$

. (C.265)

To further narrow the possible combinations of fields, we next require that the interactions in question properly descend from E_6 invariant terms of the full theory. We find that there many ways to package the field content of the MSSM into representations of E_6. The complete list of possibilities is:

	Q	U	D	L
1a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(1,2_{-1},2_1)\in 27$
1b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(1,1_{-2},2_{-1})\in \overline{27}$
1c	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
1.5	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},1_2,1_2)\in 78$	$(1,2_{-1},2_1)\in 27$
2a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
2b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(1,2_1,2_{-1})\in \overline{27}$
2.5a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(1,2_1,2_{-1})\in \overline{27}$
2.5b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(1,1_0,2_3)\in 78$
4	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_2,1_2)\in 78$	$(\overline{3},1_{-2},1_0)\in 27$	*OUT*
4.5	$(3,1_0,2_{-1})\in 27$	$(\overline{3},1_2,1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
5a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},2_1,1_0)\in 27$	*OUT*
5b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},2_1,1_0)\in 27$	$(1,1_{-2},2_{-1})\in \overline{27}$
5.3a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
5.3b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
5.6a	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_2,1_2)\in 78$	*OUT*
5.6b	$(3,1_0,2_{-1})\in 27$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_2,1_2)\in 78$	$(1,2_{-1},2_1)\in 27$
9	$(3,1_{-2},2_1)\in 78$	$(\overline{3},1_2,1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
11a	$(3,2_1,2_1)\in 78$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},2_1,1_0)\in 27$	*OUT*
11b	$(3,2_1,2_1)\in 78$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},2_1,1_0)\in 27$	$(1,2_{-1},2_1)\in 27$
11.5a	$(3,2_1,2_1)\in 78$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	*OUT*
11.5b	$(3,2_1,2_1)\in 78$	$(\overline{3},1_{-2},1_0)\in 27$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(1,1_{-2},2_1)\in \overline{27}$
13	$(3,2_1,2_1)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(\overline{3},1_{-2},1_0)\in 27$	$(1,2_1,2_{-1})\in \overline{27}$
15a	$(3,2_1,2_1)\in 78$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},2_1,1_0)\in 27$	$(1,1_0,2_3)\in 78$
15b	$(3,2_1,2_1)\in 78$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},2_1,1_0)\in 27$	$(1,2_{-1},2_1)\in 27$
15.5a	$(3,2_1,2_1)\in 78$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(1,2_{-1},2_1)\in 27$
15.5b	$(3,2_1,2_1)\in 78$	$(\overline{3},2_{-1},1_2)\in 78$	$(\overline{3},1_0,1_{-2})\in \overline{27}$	$(1,1_{-2},2_{-1})\in \overline{27}$

. (C.266)

	E	H_u	H_d	(a,b)
$1a$	$(1,1_2,1_{-2}) \in 27$	$(1,1_2,2_1) \in 27$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
$1b$	$(1,1_2,1_{-2}) \in 27$	$(1,1_2,2_1) \in 27$	$(1,1_0,2_3) \in 78$	$(2,-1)$
$1c$	$(1,2_3,1_0) \in 78$	$(1,1_2,2_1) \in 27$	$(1,1_0,2_3) \in 78$	$(2,-1)$
1.5	$(1,2_3,1_0) \in 78$	$(1,1_2,2_1) \in 27$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$2a$	$(1,1_{-2},1_2) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(1,1_0,2_3) \in 78$	$(-4,-1)$
$2b$	$(1,2_{-1},1_{-2}) \in 27$	$(1,2_{-1},2_1) \in 27$	$(1,1_0,2_3) \in 78$	$(-4,-1)$
$2.5a$	$(1,1_{-2},1_2) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(1,2_1,2_{-1}) \in \overline{27}$	$(-4,-1)$
$2.5b$	$(1,2_{-1},1_{-2}) \in 27$	$(1,2_{-1},2_1) \in 27$	$(1,2_1,2_{-1}) \in \overline{27}$	$(-4,-1)$
4	OUT	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(1,1_2,2_1) \in 27$	$(-1,-1)$
4.5	OUT	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(1,1_0,2_3) \in 78$	$(-1,-1)$
$5a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
$5b$	$(1,2_3,1_0) \in 78$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
$5.3a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_0,2_3) \in 78$	$(2,-1)$
$5.3b$	$(1,2_3,1_0) \in 78$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_0,2_3) \in 78$	$(2,-1)$
$5.6a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$5.6b$	$(1,2_3,1_0) \in 78$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
9	OUT	$(1,1_0,2_{-3}) \in 78$	$(1,1_2,2_1) \in 27$	$(-1,-1)$
$11a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$11b$	$(1,2_3,1_0) \in 78$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$11.5a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
$11.5b$	$(1,2_3,1_0) \in 78$	$(1,2_1,2_{-1}) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
13	$(1,1_{-2},1_2) \in \overline{27}$	$(1,2_{-1},2_1) \in 27$	$(1,2_1,2_{-1}) \in \overline{27}$	$(-1,2)$
$15a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,1_0,2_{-3}) \in 78$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$15b$	$(1,2_3,1_0) \in 78$	$(1,1_0,2_{-3}) \in 78$	$(1,1_{-2},2_{-1}) \in \overline{27}$	$(2,-1)$
$15.5a$	$(1,1_2,1_{-2}) \in \overline{27}$	$(1,1_0,2_{-3}) \in 78$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$
$15.5b$	$(1,2_3,1_0) \in 78$	$(1,1_0,2_{-3}) \in 78$	$(1,2_{-1},2_1) \in 27$	$(2,-1)$

(C.267)

where the numbering convention has been chosen in order to trace the origin of each possible permutation, and as before, OUT denotes an entry which has been ruled out because it cannot yield the correct $\mathrm{U}(1)_Y$ charge assignment or interaction term.

$E_6 \supset$ SU(10) $\times$ U(1). We now analyze breaking patterns of E_6 which descend from the maximal subgroup $\mathrm{SO}(10) \times \mathrm{U}(1)$ such that:

$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \tag{C.268}$$

$$27 \to 1_4 + 10_{-2} + 16_1 \tag{C.269}$$

$$78 \to 1_0 + 16_{-3} + \overline{16}_3 + 45_0. \tag{C.270}$$

Of the maximal subgroups of $SO(10)$ listed in lines (C.88)–(C.94), only the first four contain the non-abelian group $\mathrm{SU}(3)\times\mathrm{SU}(2)$ so that the unique nested sequence of maximal subgroups of E_6 is uniquely determined by the paths:

$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset [\mathrm{SU}(5) \times \mathrm{U}(1)] \times \mathrm{U}(1) \tag{C.271}$$

$$\supset [\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)] \times \mathrm{U}(1)] \times \mathrm{U}(1) \tag{C.272}$$

$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times [\mathrm{U}(1)] \tag{C.273}$$
$$\supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \times \mathrm{U}(1) \tag{C.274}$$
$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SO}(9) \times [\mathrm{U}(1)] \supset [\mathrm{SU}(2) \times \mathrm{SU}(4)] \times [\mathrm{U}(1)] \tag{C.275}$$
$$\supset [\mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)]] \times [\mathrm{U}(1)] \tag{C.276}$$
$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SO}(7) \supset [\mathrm{SU}(2) \times \mathrm{SU}(4)] \times [\mathrm{U}(1)] \tag{C.277}$$
$$\supset [\mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)]] \times [\mathrm{U}(1)]. \tag{C.278}$$

Because the previous analysis of abelian instanton configurations of SO(10) which can yield the MSSM spectrum carry over to this case as well, we focus on breaking patterns which do not embed purely in SO(10). While it is in principle possible to package the field content of the MSSM fields into representations of E_6 in more exotic ways using the additional U(1) charge, all of these configurations still correspond to generic abelian instanton configurations.

$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SO}(7) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \times [\mathrm{U}(1)]$

Decomposing the 27 and 78 with respect to this nested sequence of maximal subgroups, we find:

$$E_6 \supset \ldots \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)_a]] \times [\mathrm{U}(1)_b] \tag{C.279}$$
$$27 \to (1,1)_{0,4} + (3,1)_{0,-2} + (1,1)_{0,-2} + (1,3)_{2,-2} \tag{C.280}$$
$$+ (1,\overline{3})_{-2,-2} + (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.281}$$
$$78 \to (1,1)_{0,0} + (2,1)_{3,-3} + (2,3)_{-1,-3} + (2,1)_{-3,-3} \tag{C.282}$$
$$+ (2,\overline{3})_{1,-3} + (2,1)_{-3,3} + (2,\overline{3})_{1,3} + (2,1)_{3,3} + (2,3)_{-1,3} \tag{C.283}$$
$$+ (3,1)_{0,0} + (1,3)_{2,0} + (1,\overline{3})_{-2,0} + (3,3)_{2,0} + (3,\overline{3})_{-2,0} \tag{C.284}$$
$$+ (3,1)_{0,0} + (1,1)_{0,0} + (1,3)_{-4,0} + (1,\overline{3})_{4,0} + (1,8)_{0,0}. \tag{C.285}$$

We begin by classifying all combinations of representations which can yield the non-trivial interaction term QUH_u:

	Q	U	D	L
1	$(2,3)_{-1,1} \in 27$	$(1,\overline{3})_{-2,-2} \in 27$	$(1,\overline{3})_{-2,2} \in \overline{27}$	$(2,1)_{-3,-1} \in \overline{27}$
2	$(2,3)_{-1,1} \in 27$	$(1,\overline{3})_{-2,-2} \in 27$	$(1,\overline{3})_{4,0} \in 78$	$(2,1)_{3,-3} \in 78$
3	$(2,3)_{-1,-1} \in \overline{27}$	$(1,\overline{3})_{-2,2} \in \overline{27}$	$(1,\overline{3})_{-2,-2} \in 27$	$(2,1)_{-3,1} \in 27$
4	$(2,3)_{-1,-1} \in \overline{27}$	$(1,\overline{3})_{-2,2} \in \overline{27}$	$(1,\overline{3})_{4,0} \in 78$	$(2,1)_{3,3} \in 78$

(C.286)

	E	H_u	H_d	(a,b)
1	$(1,1)_{0,4} \in 27$	$(2,1)_{3,1} \in 27$	$(2,1)_{3,-3} \in 78$	$(1/2,3/2)$
2	$(1,1)_{0,4} \in 27$	$(2,1)_{3,1} \in 27$	$(2,1)_{-3,-1} \in \overline{27}$	$(1/2,3/2)$
3	$(1,1)_{0,-4} \in \overline{27}$	$(2,1)_{3,-1} \in \overline{27}$	$(2,1)_{3,3} \in 78$	$(1/2,-3/2)$
4	$(1,1)_{0,-4} \in \overline{27}$	$(2,1)_{3,-1} \in \overline{27}$	$(2,1)_{-3,1} \in 27$	$(1/2,-3/2)$

(C.287)

so that in this case a non-standard embedding of a $\mathrm{U}(1) \times \mathrm{U}(1)$ instanton can indeed yield the spectrum of the MSSM.

$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SO}(9) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)]] \times [\mathrm{U}(1)]$

Decomposing the 27 and 78 with respect to this nested sequence of maximal subgroups, we find:

$$E_6 \supset \ldots \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times [\mathrm{U}(1)_a]] \times [\mathrm{U}(1)_b] \tag{C.288}$$
$$27 \to (1,1)_{0,4} + (1,1)_{0,-2} + (3,1)_{0,-2} + (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} \tag{C.289}$$
$$+ (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.290}$$
$$78 \to (1,1)_{0,0} + (2,1)_{3,-3} + (2,3)_{-1,-3} + (2,1)_{-3,-3} + (2,\overline{3})_{1,-3} \tag{C.291}$$
$$+ (2,1)_{3,3} + (2,3)_{-1,3} + (2,1)_{-3,3} + (2,\overline{3})_{1,3} \tag{C.292}$$
$$+ (3,1)_{0,0} + (1,1)_{0,0} + (1,3)_{-4,0} + (1,\overline{3})_{4,0} + (1,8)_{0,0} + (3,3)_{2,0} \tag{C.293}$$
$$+ (3,\overline{3})_{-2,0} + (3,1)_{0,0} + (1,3)_{2,0} + (1,\overline{3})_{-2,0}. \tag{C.294}$$

By inspection, this is precisely the same matter content as in the previous example. We therefore conclude that the abelian instanton configurations analyzed previously produce an identical MSSM spectrum.

$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \times \mathrm{U}(1)$

The decomposition of the 27 and 78 of E_6 in this case yields:

$$E_6 \supset \mathrm{SO}(10) \times [\mathrm{U}(1)] \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times [\mathrm{U}(1)] \tag{C.295}$$
$$\supset \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times \mathrm{U}(1)_b \tag{C.296}$$
$$27 \to (1,1,1)_{0,4} + (2,2,1)_{0,-2} + (1,1,3)_{2,-2} + (1,1,\overline{3})_{-2,-2} \tag{C.297}$$
$$+ (2,1,1)_{3,1} + (2,1,3)_{-1,1} + (1,2,1)_{-3,1} + (1,2,\overline{3})_{1,1} \tag{C.298}$$
$$78 \to (1,1,1)_{0,0} + (2,1,1)_{3,-3} + (2,1,3)_{-1,-3} + (1,2,1)_{-3,-3} \tag{C.299}$$
$$+ (1,2,\overline{3})_{1,-3} + (2,1,1)_{-3,3} + (2,1,\overline{3})_{1,3} + (1,2,1)_{3,3} \tag{C.300}$$
$$+ (1,2,3)_{-1,3} + (3,1,1)_{0,0} + (1,3,1)_{0,0} + (1,1,1)_{0,0} \tag{C.301}$$
$$+ (1,1,3)_{-4,0} + (1,1,\overline{3})_{4,0} + (1,1,8)_{0,0} + (2,2,3)_{2,0} \tag{C.302}$$
$$+ (2,2,\overline{3})_{-2,0}. \tag{C.303}$$

In fact, the representation content of this decomposition is identical to that obtained via the previously treated nested sequence of maximal subgroups given by lines (C.248)–(C.255):

$$E_6 \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{U}(1)_c] \times [\mathrm{SU}(2) \times \mathrm{U}(1)_d] \tag{C.304}$$

under the linear change in U(1) charges:

$$\mathrm{U}(1)_a = \frac{1}{2}\mathrm{U}(1)_c + \frac{1}{2}\mathrm{U}(1)_d \tag{C.305}$$
$$\mathrm{U}(1)_b = \frac{1}{2}\mathrm{U}(1)_c - \frac{1}{2}\mathrm{U}(1)_d. \tag{C.306}$$

$E_6 \supset$ SU(2) $\times$ SU(6). Decomposing the 27 and 78 of E_6 with respect to SU(2) $\times$ SU(6) yields:

$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \tag{C.307}$$
$$27 \to (2,\overline{6}) + (1,15) \tag{C.308}$$
$$78 \to (3,1) + (1,35) + (2,20). \tag{C.309}$$

Returning to the maximal subgroups of SU(6) presented in lines (C.57)–(C.63), the list of all possible nested sequences of maximal subgroups of E_6 descend to G_{std} as:

$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(5) \times \mathrm{U}(1)] \tag{C.310}$$
$$\supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{SU}(2) \times [\mathrm{U}(1)] \times \mathrm{U}(1)] \tag{C.311}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1)] \tag{C.312}$$
$$\supset \mathrm{SU}(2) \times [\mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \times \mathrm{U}(1)] \tag{C.313}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.314}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.315}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.316}$$
$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(2) \times \mathrm{SU}(3)]\,. \tag{C.317}$$

In the first two nested sequences the resulting breaking pattern descends to the same representation content as breaking patterns analyzed previously. For this reason, we confine our analysis to breaking patterns reached via lines (C.314)–(C.317).

$E_6 \supset$ SU(2) $\times$ SU(6) $\supset$ SU(2) $\times$ [SU(2) $\times$ SU(3)]

In this case, the representations of E_6 decompose as:

$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(2) \times \mathrm{SU}(3)] \tag{C.318}$$
$$27 \to (2,2,\overline{3}) + (1,1,\overline{6}) + (1,3,3) \tag{C.319}$$
$$78 \to (3,1,1) + (3,3,1) + (3,3,8) + (2,4,1) + (2,2,8). \tag{C.320}$$

Decomposing one of the SU(2) factors with respect to a U(1) subgroup, it follows that the ratio of U(1) charge for the Q- and E-fields is incorrect so that the MSSM cannot be obtained via this path.

$E_6 \supset$ SU(2) $\times$ SU(6) $\supset$ SU(2) $\times$ USp(6) $\supset$ SU(2) $\times$ [SU(3) $\times$ U(1)]

The representations of E_6 descend as:

$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.321}$$
$$27 \to (2,3)_1 + (2,\overline{3})_{-1} + (1,1)_0 + (1,3)_{-2} + (1,\overline{3})_2 + (1,8)_0 \tag{C.322}$$
$$78 \to (3,1)_0 + (1,3)_{-2} + (1,\overline{3})_2 + (1,8)_0 + (1,1)_0 + (1,6)_2 \tag{C.323}$$
$$+ (1,\overline{6})_{-2} + (1,8)_0 + (2,3)_1 + (2,\overline{3})_{-1} + (2,1)_3 + (2,1)_{-3} \tag{C.324}$$
$$+ (2,6)_{-1} + (2,\overline{6})_1 \tag{C.325}$$

Because the U(1) charge assignment is incorrect, we cannot reach the MSSM via this nested sequence either.

$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)]$

Here, the representations of E_6 descend as:

$$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.326}$$

$$27 \to (2,3)_2 + (2,\overline{3})_{-2} + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 \tag{C.327}$$

$$78 \to (3,1)_0 + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 + (1,\overline{6})_{-4} \tag{C.328}$$

$$+ (1,6)_4 + (1,8)_0 + (2,3)_2 + (2,6)_{-2} + (2,1)_6 + (2,\overline{3})_{-2} \tag{C.329}$$

$$+ (2,\overline{6})_2 + (2,1)_{-6} \tag{C.330}$$

which does not contain any candidate E-fields.

$E_6 \supset \mathrm{SU}(2) \times \mathrm{SU}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)]$

All of the breaking patterns in this case have already been classified in our discussion of breaking patterns for the maximal subgroup $\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3)$. Indeed, this essentially follows from the fact that SU(3) contains the maximal subgroup $\mathrm{SU}(2)\times\mathrm{U}(1)$. We therefore proceed to the other rank six bulk gauge groups and their breaking patterns.

C.3.2 SU(7)

We assume that the matter content of SU(7) descends from the adjoint representation of E_8. For this reason, we only treat the adjoint, 7, 21, 35 and complex conjugate representations of SU(7). The maximal subgroups of SU(7) are:

$$\mathrm{SU}(7) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \tag{C.331}$$

$$\mathrm{SU}(7) \supset \mathrm{SU}(2) \times \mathrm{SU}(5) \times \mathrm{U}(1) \tag{C.332}$$

$$\mathrm{SU}(7) \supset \mathrm{SU}(3) \times \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.333}$$

$$\mathrm{SU}(7) \supset \mathrm{SO}(7) \tag{C.334}$$

of which only the first three contain G_{std}.

$\mathrm{SU}(7) \supset \mathrm{SU}(6)\times\mathrm{U}(1)$. There are three maximal subgroups of SU(6) which can contain the non-abelian factor of G_{std} and can be reached via an instanton:

$$\mathrm{SU}(7) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1) \tag{C.335}$$

$$\mathrm{SU}(7) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1) \times \mathrm{U}(1) \tag{C.336}$$

$$\mathrm{SU}(7) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{U}(1). \tag{C.337}$$

In this case, in order to preserve an $\mathrm{SU}(3) \times \mathrm{SU}(2)$ factor, the only available instanton configuration must generically take values in $\mathrm{U}(1)^3$ so that all nested sequences of maximal subgroups which can be reached by an instanton configuration all descend to the group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$. It is therefore enough to consider the breaking pattern:

$$\mathrm{SU}(7) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1) \tag{C.338}$$

$$\supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \tag{C.339}$$

$$7 \to 1_{0,0,6} + 1_{0,-5,-1} + (1,2)_{3,1,-1} + (3,1)_{-2,1,-1} \tag{C.340}$$

$$21 \to (1,1)_{6,2,-2} + (\overline{3},1)_{-4,2,-2} + (3,2)_{1,2,-2} + (3,1)_{-2,2,-2} \tag{C.341}$$
$$+ (1,2)_{3,2,-2} + (1,1)_{0,-5,5} + (3,1)_{-2,1,5} + (1,2)_{3,1,5} \tag{C.342}$$
$$35 \to (1,1)_{6,-3,-3} + (\overline{3},1)_{-4,-3,-3} + (3,2)_{1,-3,-3} + (1,1)_{-6,3,-3} \tag{C.343}$$
$$+ (3,1)_{4,3,-3} + (\overline{3},2)_{-1,3,-3} + (1,1)_{6,2,4} + (\overline{3},1)_{-4,2,4} \tag{C.344}$$
$$+ (3,2)_{1,2,4} + (3,1)_{-2,-4,4} + (1,2)_{3,-4,4} \tag{C.345}$$
$$48 \to 1_{0,0,0} + 1_{0,0,0} + (3,1)_{-2,6,0} + (1,2)_{3,6,0} + (\overline{3},1)_{2,-6,0} \tag{C.346}$$
$$+ (1,2)_{-3,-6,0} + (1,1)_{0,0,0} + (1,3)_{0,0,0} + (8,1)_{0,0,0} \tag{C.347}$$
$$+ (3,2)_{-5,0,0} + (\overline{3},2)_{5,0,0} + (1,1)_{0,-5,-7} + (1,1)_{0,5,7} \tag{C.348}$$
$$+ (3,1)_{-2,1,-7} + (1,2)_{3,1,-7} + (\overline{3},1)_{2,-1,7} + (1,2)_{-3,-1,7}. \tag{C.349}$$

By inspection, all of the representations of the MSSM are present in the above decompositions.

SU(7) ⊃ SU(2) × SU(5) × U(1). The representations of SU(7) now decompose as:

$$SU(7) \supset SU(2) \times SU(5) \times U(1) \tag{C.350}$$
$$7 \to (2,1)_5 + (1,5)_{-2} \tag{C.351}$$
$$21 \to (1,1)_{10} + (1,10)_{-4} + (2,5)_3 \tag{C.352}$$
$$35 \to (1,5)_8 + (2,10)_1 + (1,\overline{10})_{-6} \tag{C.353}$$
$$48 \to (3,1)_0 + (1,24)_0 + (2,5)_{-7} + (2,\overline{5})_7. \tag{C.354}$$

In order to retain an SU(3) subgroup, an instanton must take values in an appropriate U(1) or SU(2) subgroup of SU(5). As before, a generic $U(1)^3$ instanton will yield the expected MSSM spectrum. If we instead consider an SU(2) × U(1) instanton, it is also immediate that we can again obtain the desired spectrum of the MSSM. This alternative breaking pattern has the added benefit that it contains one less extraneous U(1) factor.

SU(7) ⊃ SU(3) × SU(4) × U(1). The representations of SU(7) decompose as:

$$SU(7) \supset SU(3) \times SU(4) \times U(1) \tag{C.355}$$
$$7 \to (3,1)_4 + (1,4)_{-3} \tag{C.356}$$
$$21 \to (\overline{3},1)_8 + (3,4)_1 + (1,6)_{-6} \tag{C.357}$$
$$35 \to (1,1)_{12} + (\overline{3},4)_5 + (3,6)_{-2} + (1,\overline{4})_{-9} \tag{C.358}$$
$$48 \to (1,1)_0 + (8,1)_0 + (1,15)_0 + (3,\overline{4})_7 + (3,4)_{-7}. \tag{C.359}$$

First suppose that the instanton configuration preserves the SU(3) subgroup of SU(4) ⊃ SU(3) × U(1). Such an instanton must then also preserve an SU(2) subgroup of the first SU(3) factor so that the resulting $U(1)^3$ instanton reduces to the generic situation treated previously.

Alternatively, an instanton can preserve all of the first SU(3) factor and break SU(4) to a smaller subgroup. To this end, recall that the maximal subgroups of SU(4) which can contain an SU(2) subgroup are:

$$\mathrm{SU}(4) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.360}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.361}$$
$$\mathrm{SU}(4) \supset \mathrm{USp}(4) \tag{C.362}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{C.363}$$

In order to preserve an SU(2) subgroup, the first case necessarily descends to the previously treated case of a $\mathrm{U}(1)^3$ instanton. We therefore focus on the remaining cases.

$\mathrm{SU}(7) \supset \mathrm{SU}(3) \times \mathrm{SU}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)] \times \mathrm{U}(1)$

In this case, we note that the resulting nested sequence of maximal subgroups descends to the same subgroup as:

$$\mathrm{SU}(7) \supset \mathrm{SU}(2) \times \mathrm{SU}(5) \times \mathrm{U}(1) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)] \times \mathrm{U}(1) \tag{C.364}$$

whose breaking patterns have already been analyzed.

$\mathrm{SU}(7) \supset \mathrm{SU}(3) \times \mathrm{SU}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times \mathrm{USp}(4) \times \mathrm{U}(1)$

Under this subgroup, the representations of SU(7) decompose as:

$$\mathrm{SU}(7) \supset \mathrm{SU}(3) \times \mathrm{SU}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times \mathrm{USp}(4) \times \mathrm{U}(1) \tag{C.365}$$
$$7 \rightarrow (3,1)_4 + (1,4)_{-3} \tag{C.366}$$
$$21 \rightarrow (\overline{3},1)_8 + (3,4)_1 + (1,1)_{-6} + (1,5)_{-6} \tag{C.367}$$
$$35 \rightarrow (1,1)_{12} + (\overline{3},4)_5 + (3,1)_{-2} + (3,5)_{-2} + (1,4)_{-9} \tag{C.368}$$
$$48 \rightarrow (1,1)_0 + (8,1)_0 + (1,5)_0 + (1,10)_0 + (3,4)_7 + (\overline{3},4)_{-7} \tag{C.369}$$

there are two possible maximal subgroups of USp(4) which can be reached by a general breaking pattern:

$$a): \mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.370}$$
$$b): \mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{U}(1). \tag{C.371}$$

We first consider the decomposition with respect to case a):

$$\mathrm{SU}(7) \supset \mathrm{SU}(3) \times \mathrm{USp}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times [\mathrm{SU}(2) \times \mathrm{SU}(2)] \times \mathrm{U}(1) \tag{C.372}$$
$$7 \rightarrow (3,1,1)_4 + (1,2,1)_{-3} + (1,1,2)_{-3} \tag{C.373}$$
$$21 \rightarrow (\overline{3},1,1)_8 + (3,2,1)_1 + (3,1,2)_1 + (1,1,1)_{-6} \tag{C.374}$$
$$+ (1,1,1)_{-6} + (1,2,2)_{-6} \tag{C.375}$$
$$35 \rightarrow (1,1,1)_{12} + (\overline{3},2,1)_5 + (\overline{3},1,2)_5 + (3,1,1)_{-2} \tag{C.376}$$
$$+ (3,1,1)_{-2} + (3,2,2)_{-2} + (1,1,2)_{-9} + (1,2,1)_{-9} \tag{C.377}$$
$$48 \rightarrow (1,1,1)_0 + (8,1,1)_0 + (1,1,1)_0 + (1,2,2)_0 \tag{C.378}$$
$$+ (1,3,1)_0 + (1,1,3)_0 + (1,2,2)_0 + (3,1,2)_7 \tag{C.379}$$
$$+ (3,2,1)_7 + (\overline{3},1,2)_{-7} + (\overline{3},2,1)_{-7}. \tag{C.380}$$

Without loss of generality, we may consider an instanton which breaks the first SU(2) factor to either the trivial group, or a U(1) subgroup. Indeed, we find that even when the instanton configuration contains a non-abelian factor, it is possible to reach the MSSM spectrum:

Q_{21}	$U_{\overline{7}}$	$D_{\overline{35}}$	L_7	$E_{\overline{21}}$	$H_{u\overline{7}}$	H_{d7}
$(3,2,1)_1$	$(3,1,1)_{-4}$	$(\overline{3},1,1)_2$	$(1,2,1)_{-3}$	$(1,1,1)_6$	$(1,2,1)_3$	$(1,2,1)_{-3}$

. (C.381)

Note that in this case, an $\mathrm{SU}(2)\times\mathrm{U}(1)$ instanton will break SU(7) directly to G_{std} with no extraneous U(1) factors.

Next consider the decomposition with respect to case b):

$$\mathrm{SU}(7)\supset\mathrm{SU}(3)\times\mathrm{USp}(4)\times\mathrm{U}(1)\supset\mathrm{SU}(3)\times[\mathrm{SU}(2)\times\mathrm{U}(1)]\times\mathrm{U}(1) \tag{C.382}$$
$$7\rightarrow(3,1)_{0,4}+(1,2)_{1,-3}+(1,2)_{-1,-3} \tag{C.383}$$
$$21\rightarrow(\overline{3},1)_{0,8}+(3,2)_{1,1}+(3,2)_{-1,1}+(1,1)_{0,-6} \tag{C.384}$$
$$+(1,1)_{2,-6}+(1,1)_{-2,-6}+(1,3)_{0,-6} \tag{C.385}$$
$$35\rightarrow(1,1)_{0,12}+(\overline{3},2)_{1,5}+(\overline{3},2)_{-1,5}+(3,1)_{0,-2} \tag{C.386}$$
$$+(3,1)_{2,-2}+(3,1)_{-2,-2}+(3,3)_{0,-2}+(1,2)_{1,-9}+(1,2)_{-1,-9} \tag{C.387}$$
$$48\rightarrow(1,1)_{0,0}+(8,1)_{0,0}+(1,1)_{2,0}+(1,1)_{2,0}+(1,1)_{-2,0} \tag{C.388}$$
$$+(1,3)_{0,0}+(1,3)_{2,0}+(1,3)_{-2,0}+(3,2)_{1,7} \tag{C.389}$$
$$+(3,2)_{-1,7}+(\overline{3},2)_{1,-7}+(\overline{3},2)_{-1,-7}. \tag{C.390}$$

In fact, with respect to the corresponding $\mathrm{U}(1)\times\mathrm{U}(1)$ instanton, we find that the matter content again organizes into the precise analogue of line (C.381) in this case as well. We therefore conclude that these candidate breaking patterns can in principle be used to eliminate additional U(1) factors.

$\mathrm{SU}(7)\supset\mathrm{SU}(3)\times\mathrm{SU}(4)\times\mathrm{U}(1)\supset\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$

The final case of interest proceeds via a different embedding of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ in SU(4) such that:

$$\mathrm{SU}(7)\supset\mathrm{SU}(3)\times\mathrm{SU}(4)\times\mathrm{U}(1)\supset\mathrm{SU}(3)\times[\mathrm{SU}(2)\times\mathrm{SU}(2)]\times\mathrm{U}(1) \tag{C.391}$$
$$7\rightarrow(3,1,1)_4+(1,2,2)_{-3} \tag{C.392}$$
$$21\rightarrow(\overline{3},1,1)_8+(3,2,2)_1+(1,1,3)_{-6}+(1,3,1)_{-6} \tag{C.393}$$
$$35\rightarrow(1,1,1)_{12}+(\overline{3},2,2)_5+(3,1,3)_{-2}+(3,3,1)_{-2}+(1,2,2)_{-9} \tag{C.394}$$
$$48\rightarrow(1,1,1)_0+(8,1,1)_0+(1,1,3)_0+(1,3,1)_0+(1,3,3)_0 \tag{C.395}$$
$$+(3,2,2)_7+(\overline{3},2,2)_{-7}. \tag{C.396}$$

Although this decomposition is indeed different from that presented below line (C.372), we note that an SU(2) instanton can generate a very similar breaking pattern. Indeed, under the forgetful homomorphism which trivializes all representations of the first SU(2) factor, we find that the two decompositions are in fact identical. In particular, this implies that a similar packaging of the field content of the MSSM as in line (C.381) will hold in this case as well.

C.3.3 SO(12)

We now proceed to the final rank six bulk gauge group which can occur in a candidate F-theory GUT model. Starting from the adjoint representation of E_8, the matter content of the bulk SO(12) theory descends from the vector 12, the spinors 32, 32′ and adjoint 66. The maximal subgroups of SO(12) are:

$$\mathrm{SO}(12) \supset \mathrm{SU}(6) \times \mathrm{U}(1) \tag{C.397}$$
$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \tag{C.398}$$
$$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \tag{C.399}$$
$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1) \tag{C.400}$$
$$\mathrm{SO}(12) \supset \mathrm{SO}(11) \tag{C.401}$$
$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SO}(9) \tag{C.402}$$
$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \tag{C.403}$$
$$\mathrm{SO}(12) \supset \mathrm{USp}(4) \times \mathrm{SO}(7). \tag{C.404}$$
$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.405}$$

of which all but the last entry contain G_{std}. As in previous examples, our expectation is that many distinct nested sequences of maximal subgroups can describe the breaking pattern of the same instanton configuration.

SO(12) ⊃ USp(4) × SO(7). The decomposition of representations of SO(12) is:

$$\mathrm{SO}(12) \supset \mathrm{USp}(4) \times \mathrm{SO}(7) \tag{C.406}$$
$$12 \to (5,1) + (1,7) \tag{C.407}$$
$$32, 32' \to (4,8) \tag{C.408}$$
$$66 \to (10,1) + (1,21) + (5,7) \tag{C.409}$$

Of the two simple group factors, only SO(7) contains an SU(3) subgroup. Further, while G_2 and SU(4) are the two maximal subgroups of SO(7) which contain an SU(3) subgroup, an instanton can only break SO(7) to SU(3) via the SU(4) path. Further decomposing with respect to the nested sequence SO(7) ⊃ SU(4) ⊃ SU(3) × U(1) therefore yields:

$$\mathrm{SO}(12) \supset \mathrm{USp}(4) \times \mathrm{SO}(7) \supset \mathrm{USp}(4) \times \mathrm{SU}(4) \supset \mathrm{USp}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.410}$$
$$12 \to (5,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.411}$$
$$32, 32' \to (4,3)_1 + (4,1)_{-3} + (4,\overline{3})_{-1} + (4,1)_3 \tag{C.412}$$
$$66 \to (10,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} + (1,3)_2 \tag{C.413}$$
$$+ (1,\overline{3})_{-2} + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 \tag{C.414}$$
$$+ (5,1)_0 + (5,3)_2 + (5,\overline{3})_{-2}. \tag{C.415}$$

With conventions as in lines (C.370) and (C.371), we now decompose USp(4) with respect to the two maximal subgroups which can break to an SU(2) factor in the presence of an SU(2) factor.

SO(12) ⊃ USp(4) × SO(7) ⊃ [SU(2) × SU(2)] × [SU(3) × U(1)$_b$]
First consider the maximal subgroup USp(4) ⊃ SU(2) × SU(2):

$$\mathrm{SO}(12) \supset \mathrm{USp}(4) \times \mathrm{SO}(7) \supset [\mathrm{SU}(2) \times \mathrm{SU}(2)] \times [\mathrm{SU}(3) \times \mathrm{U}(1)_b] \tag{C.416}$$
$$12 \to (1,1,1)_0 + (2,2,1)_0 + (1,1,1)_0 + (1,1,3)_2 + (1,1,\overline{3})_{-2} \tag{C.417}$$
$$32, 32' \to (2,1,3)_1 + (1,2,3)_1 + (2,1,1)_{-3} + (1,2,1)_{-3} + (2,1,\overline{3})_{-1} \tag{C.418}$$
$$+ (1,2,\overline{3})_{-1} + (2,1,1)_3 + (1,2,1)_3 \tag{C.419}$$
$$66 \to (3,1,1)_0 + (1,3,1)_0 + (2,2,1)_0 + (1,1,1)_0 + (1,1,3)_2 \tag{C.420}$$
$$+ (1,1,\overline{3})_{-2} + (1,1,3)_2 + (1,1,\overline{3})_{-2} + (1,1,1)_0 + (1,1,3)_{-4} \tag{C.421}$$
$$+ (1,1,\overline{3})_4 + (1,1,8)_0 + (1,1,1)_0 + (2,2,1)_0 + (1,1,3)_2 \tag{C.422}$$
$$+ (1,1,\overline{3})_{-2} + (2,2,\overline{3})_{-2}. \tag{C.423}$$

In this case it follows that an SU(2) instanton cannot yield the correct U(1)$_Y$ assignments for the fields of the MSSM. If we instead consider a U(1) instanton which breaks one of the SU(2) factors to U(1)$_a$, the following combinations of representations satisfy the requirements that all U(1)$_Y$ charge assignments are correct and further, that all interaction terms are consistent with gauge invariance of the parent theory:

	Q	U	D	L
1	$(1_0,2,3)_1$	$(1_{-1},1,\overline{3})_{-1}$	$(1_1,1,\overline{3})_{-1}$	$(1_0,2,1)_{-3}$
2	$(1_0,2,3)_1$	$(1_1,1,\overline{3})_{-1}$	$(1_{-1},1,\overline{3})_{-1}$	$(1_0,2,1)_{-3}$

(C.424)

	E	H_u	H_d	(a,b)
1	$(1_1,1,1)_3$	$(1_1,2,1)_0$	$(1_{-1},2,1)_0$	$(3,1)$
2	$(1_{-1},1,1)_3$	$(1_{-1},2,1)_0$	$(1_1,2,1)_0$	$(-3,1)$

. (C.425)

SO(12) ⊃ USp(4) × SO(7) ⊃ [SU(2) × U(1)] × [SU(3) × U(1)$_b$]
Next consider the maximal subgroup USp(4) ⊃ SU(2) × U(1):

$$\mathrm{SO}(12) \supset \mathrm{USp}(4) \times \mathrm{SO}(7) \supset [\mathrm{SU}(2) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(3) \times \mathrm{U}(1)_b] \tag{C.426}$$
$$12 \to (1_2,1_0) + (1_{-2},1_0) + (3_0,1_0) + (1_0,1_0) + (1_0,3_2) + (1_0,\overline{3}_{-2}) \tag{C.427}$$
$$32, 32' \to (2_1,3_1) + (2_{-1},3_1) + (2_1,1_{-3}) + (2_{-1},1_{-3}) + (2_1,\overline{3}_{-1}) \tag{C.428}$$
$$+ (2_{-1},\overline{3}_{-1}) + (2_1,1_3) + (2_{-1},1_3) \tag{C.429}$$
$$66 \to (1_0,1_0) + (3_0,1_0) + (3_2,1_0) + (3_{-2},1_0) + (1_0,1_0) + (1_0,3_2) \tag{C.430}$$
$$+ (1_0,\overline{3}_{-2}) + (1_0,3_2) + (1_0,\overline{3}_{-2}) + (1_0,1_0) + (1_0,3_{-4}) + (1_0,\overline{3}_4) \tag{C.431}$$
$$+ (1_0,8_0) + (1_2,1_0) + (1_{-2},1_0) + (3_0,1_0) + (1_2,3_2) + (1_{-2},3_2) \tag{C.432}$$
$$+ (3_0,3_2) + (1_2,\overline{3}_{-2}) + (1_{-2},\overline{3}_{-2}) + (3_0,\overline{3}_{-2}) \tag{C.433}$$

Listing all possible Q-, U- and H_u-fields we find:

Q	U	H_u
$(2_{\pm1},3_1)$	$(1_0,\overline{3}_{-2})$ or $(1_0,\overline{3}_4)$ or $(1_{\pm2},\overline{3}_{-2})$	$(2_{\pm1},1_{\pm3})$

. (C.434)

Note in particular that in this case, it is not possible to form a gauge invariant QUH_u, so this path is excluded.

SO(12) ⊃ SU(2) × USp(6). Because there is a unique maximal subgroup of USp(6) which contains an SU(3) factor, we may perform the unique decomposition:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{USp}(6) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.435}$$
$$12 \to (2,3)_1 + (2,\overline{3})_{-1} \tag{C.436}$$
$$32 \to (4,1)_0 + (2,3)_{-2} + (2,\overline{3})_2 + (2,8)_0 \tag{C.437}$$
$$32' \to (3,3)_1 + (3,\overline{3})_{-1} + (1,1)_3 + (1,1)_{-3} + (1,6)_{-1} + (1,\overline{6})_1 \tag{C.438}$$
$$66 \to (3,1)_0 + (1,1)_0 + (1,6)_2 + (1,\overline{6})_{-2} + (1,8)_0 + (3,3)_{-2} \tag{C.439}$$
$$+ (3,\overline{3})_2 + (3,8)_0. \tag{C.440}$$

By inspection, the relative $\mathrm{U}(1)_Y$ charge assignments for the E- and Q-fields are incorrect. We therefore conclude that this breaking pattern is not viable.

SO(12) ⊃ SU(2) × SO(9). The decomposition of SO(12) representations in this case yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SO}(9) \tag{C.441}$$
$$12 \to (3,1) + (1,9) \tag{C.442}$$
$$32, 32' \to (2,16) \tag{C.443}$$
$$66 \to (3,1) + (1,36) + (3,9). \tag{C.444}$$

There are three maximal subgroups of SO(9) which contain an SU(3) factor via a nested sequence of maximal subgroups:

$$\mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.445}$$
$$\mathrm{SO}(9) \supset \mathrm{SO}(8) \supset \mathrm{SO}(7) \supset \mathrm{SU}(4) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.446}$$
$$\mathrm{SO}(9) \supset \mathrm{SO}(8) \supset \mathrm{SU}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{U}(1) \tag{C.447}$$
$$\mathrm{SO}(9) \supset \mathrm{SO}(7) \times \mathrm{U}(1) \supset \mathrm{SU}(4) \times \mathrm{U}(1) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{U}(1). \tag{C.448}$$

By inspection, the U(1) × U(1) valued instanton associated with the last two nested sequences yield identical breaking patterns.

SO(12) ⊃ SU(2) × SO(9) ⊃ SU(2) × SU(2) × SU(4)

Decomposing the representations of SO(12) with respect to this breaking pattern yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SO}(9) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \tag{C.449}$$
$$12 \to (3,1,1) + (1,3,1) + (1,1,6) \tag{C.450}$$
$$32, 32' \to (2,2,4) + (2,2,\overline{4}) \tag{C.451}$$
$$66 \to (3,1,1) + (1,3,1) + (1,1,15) + (1,3,16) + (3,3,1) + (3,1,6) \tag{C.452}$$

In this case, the analysis of breaking patterns is similar to that of the maximal subgroup SO(10) ⊃ SU(2) × SU(2) × SU(4). We therefore conclude that the appropriate U(1) × U(1) instanton configuration can produce the spectrum of the MSSM.

SO(12) ⊃ SU(2) × SO(9) ⊃ SU(2) × SO(8) ⊃ SU(2) × SO(7)
⊃ SU(2) × SU(4) ⊃ SU(2) × [SU(3) × U(1)]

In this case, the decomposition to the appropriate subgroup does not yield a viable candidate for the E-field:

$$\mathrm{SO}(12) \supset \ldots \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \tag{C.453}$$
$$12 \to (3,1)_0 + (1,1)_0 + (1,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.454}$$
$$32, 32' \to (2,1)_3 + (2,3)_{-1} + (2,1)_{-3} + (2,\overline{3})_1 + (2,1)_3 + (2,3)_{-1} \tag{C.455}$$
$$+ (2,1)_{-3} + (2,\overline{3})_1 \tag{C.456}$$
$$66 \to (3,1)_0 + (1,1)_0 + (1,1)_0 + (1,1)_0 + (1,3)_2 + (1,\overline{3})_{-2} \tag{C.457}$$
$$+ (1,3)_2 + (1,\overline{3})_{-2} + (1,1)_0 + (1,3)_{-4} + (1,\overline{3})_4 + (1,8)_0 \tag{C.458}$$
$$+ (1,3)_2 + (1,\overline{3})_{-2} + (3,1)_0 + (3,1)_0 + (3,1)_0 + (3,3)_2 \tag{C.459}$$
$$+ (3,\overline{3})_{-2} \tag{C.460}$$

SO(12) ⊃ SU(2) × SO(9) ⊃ SU(2) × SO(8)
⊃ SU(2) × SU(4) × U(1) ⊃ SU(2) × SU(3) × U(1) × U(1)

The decomposition to G_{std} now yields:

$$\mathrm{SO}(12) \supset \ldots \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.461}$$
$$\supset \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1)_a \times \mathrm{U}(1)b \tag{C.462}$$
$$12 \to (3,1)_{0,0} + (1,1)_{0,2} + (1,1)_{0,-2} + (1,1)_{0,0} \tag{C.463}$$
$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} \tag{C.464}$$
$$32, 32' \to (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.465}$$
$$+ (2,1)_{-3,-1} + (2,\overline{3})_{1,-1} + (2,1)_{3,-1} + (2,3)_{-1,-1} \tag{C.466}$$
$$66 \to (3,1)_{0,0} + (1,1)_{0,0} + (1,1)_{0,2} + (1,3)_{2,2} \tag{C.467}$$
$$+ (1,\overline{3})_{-2,2} + (1,1)_{0,-2} + (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} \tag{C.468}$$
$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} + (1,1)_{0,0} + (1,3)_{-4,0} \tag{C.469}$$
$$+ (1,\overline{3})_{4,0} + (1,8)_{0,0} + (3,1)_{0,2} + (3,1)_{0,-2} \tag{C.470}$$
$$+ (3,1)_{0,0} + (3,3)_{2,0} + (3,\overline{3})_{-2,0}. \tag{C.471}$$

In this case, the candidate E- and Q-fields yield the relations:

$$E : \pm 2b = 6 \tag{C.472}$$
$$Q : -a \pm b = 1 \tag{C.473}$$

so that $b = \pm 3$ and $a = 2$ or -4. Because the candidate L-fields all descend from $(2,1)_{\pm 3,\pm 1}$, we further deduce that $a = 2$. Without loss of generality, we fix the sign of $b = +3$. This in turn implies that the representation content of the remaining fields is now fixed to be:

Q	U	D	L	E	H_u	H_d
$(2,3)_{-1,-1}$	$(1,\overline{3})_{-2,0}$	$(1,\overline{3})_{-2,2}$	$(2,1)_{-3,1}$	$(1,1)_{0,2}$	$(2,1)_{3,-1}$	$(2,1)_{-3,1}$

. (C.474)

Because some of the necessary interaction terms of the MSSM are now forbidden by gauge invariance of the parent theory, we conclude that this does not yield a viable breaking pattern.

SO(12) ⊃ SO(11). In this case, the breaking patterns of SO(12) directly descend to the analysis of SO(11) breaking patterns previously analyzed. Indeed, the representations of SO(12) descend as:

$$\mathrm{SO}(12) \supset \mathrm{SO}(11) \tag{C.475}$$
$$12 \to 1 + 11 \tag{C.476}$$
$$32, 32' \to 32 \tag{C.477}$$
$$66 \to 11 + 55. \tag{C.478}$$

SO(12) ⊃ SU(6) × U(1). First recall that the maximal subgroups of SU(6) which contain SU(3) × SU(2) are:

$$\mathrm{SU}(6) \supset \mathrm{SU}(5) \times \mathrm{U}(1) \tag{C.479}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.480}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{C.481}$$
$$\mathrm{SU}(6) \supset \mathrm{SU}(2) \times \mathrm{SU}(3). \tag{C.482}$$

In the first three cases we find that the resulting breaking pattern must descend to the usual breaking pattern via a $\mathrm{U}(1)^3$ instanton. Finally, by inspection of the decomposition of SO(12) ⊃ SU(6) × U(1), we note that the resulting integral U(1) charges of each decomposition are bounded in magnitude by two. Hence, only the first three maximal subgroups can yield a consistent breaking pattern. While it would be of interest to classify all possible ways of packaging the field content of the MSSM in representations of SO(12) in this case, this analysis is not necessary for the purposes of classifying breaking patterns.

SO(12) ⊃ SU(2)×SU(2)×SO(8). Decomposing all relevant representations of SO(12) with respect to this maximal subgroup yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \tag{C.483}$$
$$12 \to (2,2,1) + (1,1,8^v) \tag{C.484}$$
$$32 \to (1,2,8^s) + (2,1,8^c) \tag{C.485}$$
$$32' \to (1,2,8^c) + (2,1,8^s) \tag{C.486}$$
$$66 \to (3,1,1) + (1,3,1) + (1,1,28) + (2,2,8^v). \tag{C.487}$$

There are two maximal subgroups of SO(8) which are consistent with a breaking pattern generated by an instanton configuration:

$$\mathrm{SO}(8) \supset \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.488}$$
$$\mathrm{SO}(8) \supset \mathrm{SO}(7) \supset \mathrm{SU}(4). \tag{C.489}$$

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We now consider breaking patterns which can descend from both maximal subgroups.

SO(12) ⊃ SU(2) × SU(2) × SO(8) ⊃ SU(2) × SU(2) × [SU(4) × U(1)]

Because the only simple group factor which contains an SU(3) subgroup is SU(4), we may further decompose SU(4) ⊃ SU(3) × U(1). This yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(8) \supset \mathrm{SU}(2)\times\mathrm{SU}(2)\times[\mathrm{SU}(4)\times\mathrm{U}(1)] \tag{C.490}$$
$$\supset \mathrm{SU}(2)\times\mathrm{SU}(2)\times[\mathrm{SU}(3)\times\mathrm{U}(1)\times\mathrm{U}(1)] \tag{C.491}$$
$$12 \to (2,2,1)_{0,0} + (1,1,1)_{0,2} + (1,1,1)_{0,-2} \tag{C.492}$$
$$+ (1,1,3)_{2,0} + (1,1,\overline{3})_{-2,0} \tag{C.493}$$
$$32 \to (1,2,1)_{3,1} + (1,2,3)_{-1,1} + (1,2,1)_{-3,-1} \tag{C.494}$$
$$+ (1,2,\overline{3})_{1,-1} + (2,1,1)_{3,-1} + (2,1,3)_{-1,-1} \tag{C.495}$$
$$+ (2,1,1)_{-3,1} + (2,1,\overline{3})_{1,1} \tag{C.496}$$
$$32' \to (2,1,1)_{3,1} + (2,1,3)_{-1,1} + (2,1,1)_{-3,-1} \tag{C.497}$$
$$+ (2,1,\overline{3})_{1,-1} + (1,2,1)_{3,-1} + (1,2,3)_{-1,-1} \tag{C.498}$$
$$+ (1,2,1)_{-3,1} + (1,2,\overline{3})_{1,1} \tag{C.499}$$
$$66 \to (3,1,1)_{0,0} + (1,3,1)_{0,0} + (1,1,1)_{0,0} \tag{C.500}$$
$$+ (1,1,3)_{2,2} + (1,1,\overline{3})_{-2,2} + (1,1,3)_{2,-2} \tag{C.501}$$
$$+ (1,1,\overline{3})_{-2,-2} + (1,1,1)_{0,0} + (1,1,3)_{-4,0} \tag{C.502}$$
$$+ (1,1,\overline{3})_{4,0} + (1,1,8)_{0,0} + (2,2,1)_{0,2} \tag{C.503}$$
$$+ (2,2,1)_{0,-2} + (2,2,3)_{2,0} + (2,2,\overline{3})_{-2,0}\,. \tag{C.504}$$

If we now consider a U(1) instanton which breaks one of the SU(2) factor, we again obtain a $\mathrm{U}(1)^3$ instanton configuration. Indeed, this case is quite similar to breaking via the maximal subgroup SU(2) × SU(2) × SU(4) ⊂ SO(10) considered previously.

Next suppose without loss of generality that an instanton configuration takes values in the first SU(2) factor such that it breaks either to U(1) or trivial group. Because the abelian case is quite similar, we assume that the non-abelian instanton breaks all of SU(2). In this case, the list of candidate Q-, U- and H_u-fields which can yield a gauge invariant QUH_u interaction are:

	Q	U	H_u	(a,b)
1	$(1,2,3)_{-1,1}$	$(1,1,\overline{3})_{-2,0}$	$(1,2,1)_{3,-1}$	$(2,3)$
2	$(1,2,3)_{-1,1}$	$(2,1,\overline{3})_{1,1}$	$(2,2,1)_{0,-2}$	$(-5/2,-3/2)$
3	$(1,2,3)_{-1,1}$	$(2,1,\overline{3})_{1,-1}$	OUT	OUT
4	$(1,2,3)_{-1,-1}$	$(1,1,\overline{3})_{-2,0}$	$(1,2,1)_{3,1}$	$(2,-3)$
5	$(1,2,3)_{-1,-1}$	$(2,1,\overline{3})_{1,1}$	OUT	OUT
6	$(1,2,3)_{-1,-1}$	$(2,1,\overline{3})_{1,-1}$	$(2,2,1)_{0,2}$	$(-5/2,3/2)$
7	$(2,2,3)_{2,0}$	$(1,1,\overline{3})_{-2,0}$	OUT	OUT
8	$(2,2,3)_{2,0}$	$(2,1,\overline{3})_{1,1}$	$(1,2,1)_{-3,-1}$	$(1/2,-9/2)$
9	$(2,2,3)_{2,0}$	$(2,1,\overline{3})_{1,-1}$	$(1,2,1)_{-3,1}$	$(1/2,9/2)$

. (C.505)

Restricting to the six viable remaining possibilities, we now find that no candidate D-field reproduces the correct $\mathrm{U}(1)_Y$ charge assignment. We therefore conclude that only abelian instanton configurations can yield the spectrum of the MSSM in this case.

$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(7)$

$\supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)]$

Along this nested sequence of maximal subgroups, the decomposition of the representations of SO(12) is:

$$\begin{aligned}
\mathrm{SO}(12) &\supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(7) && \text{(C.506)}\\
&\supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)_b] && \text{(C.507)}\\
12 &\to (2,2,1)_0 + (1,1,1)_0 + (1,1,1)_0 + (1,1,3)_2 + (1,1,\overline{3})_{-2} && \text{(C.508)}\\
32, 32' &\to (1,2,1)_3 + (1,2,3)_{-1} + (1,2,1)_{-3} + (1,2,\overline{3})_1 + (2,1,1)_3 && \text{(C.509)}\\
&+ (2,1,3)_{-1} + (2,1,1)_{-3} + (2,1,\overline{3})_1 && \text{(C.510)}\\
66 &\to (3,1,1)_0 + (1,3,1)_0 + (1,1,1)_0 + (1,1,3)_2 + (1,1,\overline{3})_{-2} && \text{(C.511)}\\
&+ (1,1,3)_2 + (1,1,\overline{3})_{-2} + (1,1,1)_0 + (1,1,3)_{-4} + (1,1,\overline{3})_4 && \text{(C.512)}\\
&+ (1,1,8)_0 + (2,2,1)_0 + (2,2,1)_0 + (2,2,3)_2 + (2,2,\overline{3})_{-2}. && \text{(C.513)}
\end{aligned}$$

By inspection, the above $\mathrm{U}(1)_b$ charge assignments do not agree with those of the MSSM. It thus follows that we must further break one of the SU(2) factors to U(1). Without loss of generality, we assume that the first SU(2) factor decomposes further to a maximal $\mathrm{U}(1)_a$ subgroup. The list of candidate Q-, U- and H_u-fields which can yield a gauge invariant QUH_u interaction are therefore:

	Q	U	H_u	(a,b)
1	$(1_0,2,3)_{-1}$	$(1_0,1,\overline{3})_{-2}$	$(1_0,2,1)_3$	*OUT*
2	$(1_0,2,3)_{-1}$	$(1_1,1,\overline{3})_1$	$(1_{-1},2,1)_0$	$(-3,-1)$
3	$(1_0,2,3)_{-1}$	$(1_{-1},1,\overline{3})_1$	$(1_1,2,1)_0$	$(3,-1)$
4	$(1_0,2,3)_{-1}$	$(1_0,1,\overline{3})_4$	$(1_0,2,1)_{-3}$	$(a,-1)$
5	$(1_1,2,3)_2$	$(1_0,1,\overline{3})_{-2}$	$(1_{-1},2,1)_0$	$(-3,2)$
6	$(1_1,2,3)_2$	$(1_1,1,\overline{3})_1$	*OUT*	*OUT*
7	$(1_1,2,3)_2$	$(1_{-1},1,\overline{3})_1$	$(1_0,2,1)_{-3}$	$(3,-1)$
8	$(1_1,2,3)_2$	$(1_0,1,\overline{3})_4$	*OUT*	*OUT*
9	$(1_{-1},2,3)_2$	$(1_0,1,\overline{3})_{-2}$	$(1_1,2,1)_0$	$(3,2)$
10	$(1_{-1},2,3)_2$	$(1_1,1,\overline{3})_1$	$(1_0,2,1)_{-3}$	$(-3,-1)$
11	$(1_{-1},2,3)_2$	$(1_{-1},1,\overline{3})_1$	*OUT*	*OUT*
12	$(1_{-1},2,3)_2$	$(1_0,1,\overline{3})_4$	*OUT*	*OUT*

. (C.514)

Next, we list all candidate D- and H_d-fields which can yield a gauge invariant QDH_d

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interaction term:

	Q	D	H_d	(a,b)
$2a$	$(1_0,2,3)_{-1}$	$(1_0,1,\overline{3})_{-2}$	$(1_0,2,1)_3$	$(-3,-1)$
$2b$	$(1_0,2,3)_{-1}$	$(1_{-1},1,\overline{3})_1$	$(1_1,2,1)_0$	$(-3,-1)$
$3a$	$(1_0,2,3)_{-1}$	$(1_0,1,\overline{3})_{-2}$	$(1_0,2,1)_3$	$(3,-1)$
$3b$	$(1_0,2,3)_{-1}$	$(1_1,1,\overline{3})_1$	$(1_{-1},2,1)_0$	$(3,-1)$
$4a$	$(1_0,2,3)_{-1}$	$(1_0,1,\overline{3})_{-2}$	$(1_0,2,1)_3$	$(a,-1)$
$4b$	$(1_0,2,3)_{-1}$	$(1_1,1,\overline{3})_1$	$(1_{-1},2,1)_0$	$(3,-1)$
$4c$	$(1_0,2,3)_{-1}$	$(1_{-1},1,\overline{3})_1$	$(1_1,2,1)_0$	$(-3,-1)$
5	$(1_1,2,3)_2$	*OUT*	*OUT*	$(-3,2)$
$7a$	$(1_1,2,3)_2$	$(1_0,1,\overline{3})_{-2}$	$(1_{-1},2,1)_0$	$(3,-1)$
$7b$	$(1_1,2,3)_2$	$(1_1,1,\overline{3})_1$	*OUT*	$(3,-1)$
9	$(1_{-1},2,3)_2$	$(1_1,1,\overline{3})_1$	*OUT*	$(3,2)$
$10a$	$(1_{-1},2,3)_2$	$(1_0,1,\overline{3})_{-2}$	$(1_1,2,1)_0$	$(-3,-1)$
$10b$	$(1_{-1},2,3)_2$	$(1_{-1},1,\overline{3})_1$	*OUT*	$(-3,-1)$

. (C.515)

Of these remaining possibilities, we now determine all possible candidate L- and E-fields which can yield the gauge invariant interaction term LEH_d:

	L	E	H_d	(a,b)
$2a$	$(1_1,2,1)_0$	$(1_{-1},1,1)_{-3}$	$(1_0,2,1)_3$	$(-3,-1)$
$2b$	$(1_0,2,1)_3$	$(1_{-1},1,1)_{-3}$	$(1_1,2,1)_0$	$(-3,-1)$
$2b'$	$(1_1,2,1)_0$	$(1_{-2},1,1)_0$	$(1_1,2,1)_0$	$(-3,-1)$
$3a$	$(1_{-1},2,1)_0$	$(1_1,1,1)_{-3}$	$(1_0,2,1)_3$	$(3,-1)$
$3b$	$(1_{-1},2,1)_0$	$(1_2,1,1)_0$	$(1_{-1},2,1)_0$	$(3,-1)$
$3b'$	$(1_0,2,1)_3$	$(1_1,1,1)_{-3}$	$(1_{-1},2,1)_0$	$(3,-1)$
$4a$	$(1_{\pm1},2,1)_0$	$(1_{\mp1},1,1)_{-3}$	$(1_0,2,1)_3$	$(\mp3,-1)$
$4b$	$(1_{-1},2,1)_0$	$(1_2,1,1)_0$	$(1_{-1},2,1)_0$	$(3,-1)$
$4b'$	$(1_0,2,1)_3$	$(1_1,1,1)_{-3}$	$(1_{-1},2,1)_0$	$(3,-1)$
$4c$	$(1_1,2,1)_0$	$(1_{-2},1,1)_0$	$(1_1,2,1)_0$	$(-3,-1)$
$4c'$	$(1_0,2,1)_3$	$(1_{-1},1,1)_{-3}$	$(1_1,2,1)_0$	$(-3,-1)$
$7a$	$(1_{-1},2,1)_0$	$(1_2,1,1)_0$	$(1_{-1},2,1)_0$	$(3,-1)$
$7a'$	$(1_0,2,1)_3$	$(1_1,1,1)_{-3}$	$(1_{-1},2,1)_0$	$(3,-1)$
$10a$	$(1_1,2,1)_0$	$(1_{-2},1,1)_0$	$(1_1,2,1)_0$	$(-3,-1)$
$10a'$	$(1_0,2,1)_3$	$(1_{-1},1,1)_{-3}$	$(1_1,2,1)_0$	$(-3,-1)$

. (C.516)

Note that in this case, there are many distinct ways to package the field content of the MSSM such that SO(12) breaks to SU(3) × SU(2) × U(1) × U(1) via a $U(1)^2$ instanton configuration.

SO(12) ⊃ SU(4) × SU(4). Decomposing representations of SO(12) with respect to the maximal subgroup SU(4) × SU(4) yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \tag{C.517}$$
$$12 \to (6,1) + (1,6) \tag{C.518}$$
$$12 \to (6,1) + (1,6) \tag{C.519}$$
$$32' \to (4,\overline{4}) + (\overline{4},4) \tag{C.520}$$
$$66 \to (15,1) + (1,15) + (6,6). \tag{C.521}$$

Without loss of generality, we assume that the first SU(4) factor further breaks to SU(3) × U(1). The remaining nested sequences of maximal subgroups which can yield the Standard Model gauge group are:

$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.522}$$
$$\mathrm{SU}(4) \supset \mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.523}$$
$$\mathrm{SU}(4) \supset \mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{U}(1) \tag{C.524}$$
$$\mathrm{SU}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2). \tag{C.525}$$

$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$

In this case, it follows at once from the local isomorphisms $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2) \simeq \mathrm{SO}(4)$ that the endpoint of this breaking pattern is identical to the endpoint of the nested sequence of maximal subgroups:

$$\mathrm{SO}(12) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SO}(8) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1) \tag{C.526}$$
$$\supset \mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)] \times \mathrm{U}(1). \tag{C.527}$$

We therefore conclude that all breaking patterns via instantons have in this case been catalogued.

$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times \mathrm{USp}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$

The decomposition of the representations of SO(12) with respected to this sequence of maximal subgroups is:

$$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times \mathrm{USp}(4) \tag{C.528}$$
$$\supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(2) \times \mathrm{SU}(2)] \tag{C.529}$$
$$12 \to (3,1,1)_2 + (\overline{3},1,1)_{-2} + (1,1,1)_0 + (1,1,1)_0 + (1,2,2)_0 \tag{C.530}$$
$$32, 32' \to (1,2,1)_3 + (1,1,2)_3 + (3,2,1)_{-1} + (3,1,2)_{-1} + (1,2,1)_{-3} \tag{C.531}$$
$$+ (1,1,2)_{-3} + (\overline{3},2,1)_1 + (\overline{3},1,2)_1 \tag{C.532}$$
$$66 \to (1,1,1)_0 + (3,1,1)_{-4} + (\overline{3},1,1)_4 + (8,1,1)_0 + (1,1,1)_0 \tag{C.533}$$
$$+ (1,2,2)_0 + (1,3,1)_0 + (1,1,3)_0 + (1,2,2)_0 + (3,1,1)_2 \tag{C.534}$$
$$+ (\overline{3},1,1)_{-2} + (3,1,1)_2 + (3,2,2)_2 + (\overline{3},1,1)_{-2} + (\overline{3},2,2)_{-2}. \tag{C.535}$$

By inspection of the above representation content, we note that while an SU(2) instanton which breaks either of the SU(2) factors could yield the correct gauge group, the resulting $\mathrm{U}(1)_Y$ charge assignments of the fields would be incorrect. It is therefore enough to consider abelian instanton configurations which break one of the SU(2) factors to $\mathrm{U}(1)_b$. Due to the symmetry between the two SU(2) factors, we assume without loss of generality that the instanton preserves the first SU(2) factor. We begin by listing the candidate representations for the Q-, U- and H_u- fields which can yield the interaction term QUH_u as well as the correct $\mathrm{U}(1)_Y$ charge assignments:

	Q	U	H_u	(a,b)
1	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_{\pm 1})_1$	$(1,2,1_{\mp 1})_0$	$(\mp 3,-1)$
2	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_0)_4$	$(1,2,1_0)_{-3}$	$(a,-1)$
3	$(3,2,1_{\pm 1})_2$	$(\overline{3},1,1_0)_{-2}$	$(1,2,1_{\mp 1})_0$	$(\mp 3,2)$
4	$(3,2,1_{\pm 1})_2$	$(\overline{3},1,1_{\mp 1})_1$	$(1,2,1_0)_{-3}$	$(\pm 3,-1)$

(C.536)

where in the above, all $\pm$'s of a given row are correlated. Of these four possibilities, we now list all candidate representations for the D- and H_d-fields which can yield the interaction term QDH_d:

	Q	D	H_d	(a,b)
$1a$	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_{\mp 1})_1$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$1b$	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_0)_{-2}$	$(1,2,1_0)_3$	$(\mp 3,-1)$
$2a$	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_{\mp 1})_1$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$2b$	$(3,2,1_0)_{-1}$	$(\overline{3},1,1_0)_{-2}$	$(1,2,1_0)_3$	$(a,-1)$
4	$(3,2,1_{\pm 1})_2$	$(\overline{3},1,1_0)_{-2}$	$(1,2,1_{\mp 1})_0$	$(\pm 3,-1)$

. (C.537)

Finally, we list all candidate E- and L- fields which can yield the term ELH_d:

	E	L	H_d	(a,b)
$1a$	$(1,1,1_{\mp 1})_{-3}$	$(1,2,1_0)_3$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$1a'$	$(1,1,1_{\mp 2})_0$	$(1,2,1_{\pm 1})_0$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$1b$	$(1,1,1_{\mp 1})_{-3}$	$(1,2,1_{\pm 1})_0$	$(1,2,1_0)_3$	$(\mp 3,-1)$
$2a$	$(1,1,1_{\mp 1})_{-3}$	$(1,2,1_0)_3$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$2a'$	$(1,1,1_{\mp 2})_0$	$(1,2,1_{\pm 1})_0$	$(1,2,1_{\pm 1})_0$	$(\mp 3,-1)$
$2b$	$(1,1,1_{\mp 1})_{-3}$	$(1,2,1_{\pm 1})_0$	$(1,2,1_0)_3$	$(\mp 3,-1)$
4	$(1,1,1_{\pm 1})_{-3}$	$(1,2,1_0)_3$	$(1,2,1_{\mp 1})_0$	$(\pm 3,-1)$
$4'$	$(1,1,1_{\pm 2})_0$	$(1,2,1_{\mp 1})_0$	$(1,2,1_{\mp 1})_0$	$(\pm 3,-1)$

. (C.538)

We note that in this case, while there are only two linear combinations of the two U(1) factors which can yield $\mathrm{U}(1)_Y$, there are different ways to package the fields of the MSSM in representations of SO(12).

$\mathrm{SO}(12) \supset \mathrm{SU}(4)\times\mathrm{SU}(4) \supset [\mathrm{SU}(3)\times\mathrm{U}(1)_a]\times\mathrm{USp}(4) \supset [\mathrm{SU}(3)\times\mathrm{U}(1)_a]\times[\mathrm{SU}(2)\times\mathrm{U}(1)_b]$

In this case, the decomposition of representations of SO(12) yields:

$$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times \mathrm{USp}(4) \tag{C.539}$$

$$\supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(2) \times \mathrm{U}(1)_b] \tag{C.540}$$

$$12 \to (3_2, 1_0) + (\overline{3}_{-2}, 1_0) + (1_0, 1_0) + (1_0, 1_2) + (1_0, 1_{-2}) + (1_0, 3_0) \tag{C.541}$$

$$32, 32' \to (1_3, 2_1) + (1_3, 2_{-1}) + (3_{-1}, 2_1) + (3_{-1}, 2_{-1}) \tag{C.542}$$
$$+ (1_{-3}, 2_1) + (1_{-3}, 2_{-1}) + (\overline{3}_1, 2_1) + (\overline{3}_1, 2_{-1}) \tag{C.543}$$
$$66 \to (1_0, 1_0) + (3_{-4}, 1_0) + (\overline{3}_4, 1_0) + (8_0, 1_0) + (1_0, 1_2) + (1_0, 1_{-2}) \tag{C.544}$$
$$+ (1_0, 3_0) + (1_0, 1_0) + (1_0, 3_0) + (1_0, 3_2) + (1_0, 3_{-2}) \tag{C.545}$$
$$+ (3_2, 1_0) + (\overline{3}_{-2}, 1_0) + (3_2, 1_2) + (3_2, 1_{-2}) + (3_2, 3_0) \tag{C.546}$$
$$+ (\overline{3}_{-2}, 1_2) + (\overline{3}_{-2}, 1_{-2}) + (\overline{3}_{-2}, 3_0). \tag{C.547}$$

We note in passing that this indeed yields a distinct decomposition from the previous breaking pattern. By inspection, the only candidate E-fields are $(1_0, 1_{\pm 2})$ so that $b = \pm 3$. Listing all Q-, U- and H_u-fields which can yield a gauge invariant interaction term QUH_u such that $b = \pm 3$ is indeed a solution, we find:

Q	U	H_u	(a, b)
$(3_{-1}, 2_{\pm 1})$	$(\overline{3}_{-2}, 1_0)$	$(1_3, 2_{\mp 1})$	$(2, \pm 3)$

(C.548)

where all $\pm$'s in a given row are correlated. Listing all Q-, D- and H_d- fields which can yield the term QDH_d, we find:

Q	D	H_d	(a, b)
$(3_{-1}, 2_{\pm 1})$	$(\overline{3}_{-2}, 1_{\mp 2})$	$(1_{-3}, 2_{\pm 1})$	$(2, \pm 3)$

. (C.549)

Now, we find that in this case, the only candidate L- and H_d-fields are $(1_{-3}, 2_{\pm 1})$. In particular, it follows that the purported ELH_d interaction will violate $\mathrm{U}(1)_a$ because the only candidate E-field is neutral under $\mathrm{U}(1)_a$ so that this breaking pattern cannot yield the spectrum of the MSSM.

$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(2) \times \mathrm{SU}(2)]$

In this case, the decomposition of the representations of SO(12) is given by:

$$\mathrm{SO}(12) \supset \mathrm{SU}(4) \times \mathrm{SU}(4) \supset [\mathrm{SU}(3) \times \mathrm{U}(1)_a] \times [\mathrm{SU}(2) \times \mathrm{SU}(2)] \tag{C.550}$$
$$12 \to (3_2, 1, 1) + (\overline{3}_{-2}, 1, 1) + (1_0, 3, 1) + (1_0, 1, 3) \tag{C.551}$$
$$32, 32' \to (1_3, 2, 2) + (3_{-1}, 2, 2) + (1_{-3}, 2, 2) + (\overline{3}_1, 2, 2) \tag{C.552}$$
$$66 \to (1_0, 1, 1) + (3_{-4}, 1, 1) + (\overline{3}_4, 1, 1) + (8_0, 1, 1) + (1_0, 1, 3) \tag{C.553}$$
$$+ (1_0, 3, 1) + (1_0, 3, 3) + (3_2, 1, 3) + (3_2, 3, 1) + (\overline{3}_{-2}, 1, 3) \tag{C.554}$$
$$+ (\overline{3}_{-2}, 3, 1). \tag{C.555}$$

By inspection, we must consider an abelian instanton configuration which breaks one of the SU(2) factors to a $\mathrm{U}(1)_b$ subgroup. Without loss of generality, we assume that the instanton preserves the first SU(2) factor. In this case, the resulting candidate E-fields are all of the form $(1_0, 1, 1_{\pm 2})$ so that $b = \pm 3$. Listing all candidate Q-, U- and H_u-fields which can yield a gauge invariant term of the form QUH_u, we find:

Q	U	H_u	(a, b)
$(3_{-1}, 2, 1_{\pm 1})$	$(\overline{3}_{-2}, 1, 1_0)$	$(1_3, 2_{\mp 1})$	$(2, \pm 3)$

(C.556)

where all ±'s are correlated in the above. This in turn implies that there is a unique candidate H_d-field given by $(1_{-3}, 2_{\pm 1})$. This in turn requires that in order to obtain a non-zero QDH_d interaction term, a candidate D-field must have representation content $(\overline{3}_4, 1, 1_{\mp 2})$ which is not present in the given decomposition described above. We therefore conclude that this breaking pattern cannot yield the spectrum of the MSSM.

SO(12) ⊃ SO(10) × U(1). This is the final maximal subgroup of SO(12) which can in principle contain $G_{\rm std}$. The representation content of SO(12) decomposes under this maximal subgroup as:

$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1) \tag{C.557}$$
$$12 \to 1_2 + 1_{-2} + 10_0 \tag{C.558}$$
$$32 \to 16_1 + \overline{16}_{-1} \tag{C.559}$$
$$32' \to \overline{16}_1 + 16_{-1} \tag{C.560}$$
$$66 \to 1_0 + 10_2 + 10_{-2} + 45_0. \tag{C.561}$$

Recall that the maximal subgroups of SO(10) are listed in lines (C.88)–(C.94), of which only lines (C.88)-(C.91) contain an SU(3) × SU(2) subgroup. In the present context, we wish to determine whether the presence of the additional U(1) factor can yield a new breaking pattern distinct from those already treated for $G_S = \mathrm{SO}(10)$. Moreover, while it is in principle of interest to classify all ways of packaging the fields of the MSSM into SO(12) representations, our primary interest is in the classification of all possible breaking patterns. For this reason, we again confine our classification to this more narrow question.

SO(12) ⊃ SO(10) × U(1) ⊃ SU(5) × U(1) × U(1)

In this case, there is a unique way in which the SU(5) factor can further break to $G_{\rm std}$. Indeed, this is the natural extension of the analogous breaking pattern of SO(10) analyzed previously. We thus conclude that in this case the abelian $\mathrm{U}(1)^3$ instanton breaks SO(12) to $G_{\rm std} \times \mathrm{U}(1) \times \mathrm{U}(1)$.

SO(12) ⊃ SO(10) × U(1) ⊃ [SU(2) × SU(2) × SU(4)] × U(1)

Under this nested sequence of maximal subgroups, SU(4) is the only factor which contains an SU(3) subgroup. The representation content of SO(12) therefore must decompose as:

$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1) \supset [\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)] \times \mathrm{U}(1)_b \tag{C.562}$$
$$\supset [\mathrm{SU}(2) \times \mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)_a]] \times \mathrm{U}(1)_b \tag{C.563}$$
$$12 \to (1,1,1)_{0,2} + (1,1,1)_{0,-2} + (2,2,1)_{0,0} + (1,1,3)_{2,0} + (1,1,\overline{3})_{-2,0} \tag{C.564}$$
$$32 \to (2,1,1)_{3,1} + (2,1,3)_{-1,1} + (1,2,1)_{-3,1} + (1,2,\overline{3})_{1,1} \tag{C.565}$$
$$+ (2,1,1)_{-3,-1} + (2,1,\overline{3})_{1,-1} + (1,2,1)_{3,-1} + (1,2,3)_{-1,-1} \tag{C.566}$$
$$32' \to (1,2,1)_{3,1} + (1,2,3)_{-1,1} + (2,1,1)_{-3,1} + (2,1,\overline{3})_{1,1} \tag{C.567}$$
$$+ (1,2,1)_{-3,-1} + (1,2,\overline{3})_{1,-1} + (2,1,1)_{3,-1} + (2,1,3)_{-1,-1} \tag{C.568}$$
$$66 \to (1,1,1)_{0,0} + (2,2,1)_{0,2} + (1,1,3)_{2,2} + (1,1,\overline{3})_{-2,2} \tag{C.569}$$
$$+ (2,2,1)_{0,-2} + (1,1,3)_{2,-2} + (1,1,\overline{3})_{-2,-2} + (3,1,1)_{0,0} \tag{C.570}$$
$$+ (1,3,1)_{0,0} + (1,1,1)_{0,0} + (1,1,3)_{-4,0} + (1,1,\overline{3})_{4,0} \tag{C.571}$$
$$+ (1,1,8)_{0,0} + (2,2,3)_{2,0} + (2,2,\overline{3})_{-2,0}. \tag{C.572}$$

In the present context, breaking one of the SU(2) factors to a U(1) subgroup yields a breaking pattern identical to that already studied in the context of the sequence of maximal subgroups $\mathrm{SO}(12) \supset \mathrm{SO}(10)\times \mathrm{U}(1) \supset \mathrm{SU}(5)\times \mathrm{U}(1)\times \mathrm{U}(1) \supset \mathrm{SU}(3)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{U}(1)$. In order to classify all candidate breaking patterns, it is therefore enough to restrict to cases where one of the SU(2) factors is completely broken. Without loss of generality, we assume that the candidate non-abelian instanton preserves the second SU(2) factor. Listing all candidate Q-, U- and H_u-fields which can yield the gauge invariant interaction term QUH_u, we find:

	Q	U	H_u	(a,b)
1	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{-2,0}$	$(1,2,1)_{3,\mp 1}$	$(2,\pm 3)$
2	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{4,0}$	$(1,2,1)_{-3,\mp 1}$	$(-1,0)$
3	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{-2,\mp 2}$	$(1,2,1)_{3,\pm 1}$	$(1/2,\pm 3/2)$
4	$(1,2,3)_{-1,\pm 1}$	$(2,1,\overline{3})_{1,\pm 1}$	$(2,2,1)_{0,\mp 2}$	$(-5/2,\mp 3/2)$
5	$(2,2,3)_{2,0}$	$(2,1,\overline{3})_{1,\pm 1}$	$(1,2,1)_{-3,\mp 1}$	$(1/2,\mp 9/2)$
6	$(2,2,3)_{2,0}$	$(1,1,\overline{3})_{-2,\mp 2}$	$(2,2,1)_{0,\pm 2}$	$(1/2,\pm 3/2)$

. (C.573)

Listing all choices of representations for candidate D- and H_d-fields which also admit the gauge invariant interaction term QDH_d, we find:

	Q	D	H_d	(a,b)
$2a$	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{-2,0}$	$(1,2,1)_{3,\mp 1}$	$(-1,0)$
$2b$	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{-2,\mp 2}$	$(1,2,1)_{3,\pm 1}$	$(-1,0)$
$3a$	$(1,2,3)_{-1,\pm 1}$	$(1,1,\overline{3})_{4,0}$	$(1,2,1)_{-3,\mp 1}$	$(1/2,\pm 3/2)$
$3b$	$(1,2,3)_{-1,\pm 1}$	$(2,1,\overline{3})_{1,\pm 1}$	$(2,2,1)_{0,\mp 2}$	$(1/2,\pm 3/2)$
$6a$	$(2,2,3)_{2,0}$	$(1,1,\overline{3})_{-2,\pm 2}$	$(2,2,1)_{0,\mp 2}$	$(1/2,\pm 3/2)$
$6b$	$(2,2,3)_{2,0}$	$(2,1,\overline{3})_{1,\pm 1}$	$(1,2,1)_{-3,\mp 1}$	$(1/2,\pm 3/2)$

. (C.574)

Because the only candidate E-fields are given by $(1,1,1)_{0,\pm 2}$ or $(1,2,1)_{\pm 3,\pm 1}$, we now observe that all consistent choices of $\mathrm{U}(1)_Y$ given previously cannot yield the correct value for the E-fields. Hence, an instanton configuration must break one of the SU(2) factors to a U(1) subgroup in order to reproduce the spectrum of the MSSM.

$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1) \supset \mathrm{SO}(9) \times \mathrm{U}(1)$

In order to obtain an $\mathrm{SU}(3) \times \mathrm{SU}(2)$ subgroup along this nested sequence of maximal subgroups, the SO(9) factor must also contain such a subgroup. Returning to lines (C.36)–(C.40), we again conclude that the only maximal subgroup of SO(9) satisfying this criterion is $\mathrm{SU}(2)\times \mathrm{SU}(4)$. Further decomposing the SU(4) factor to the maximal subgroup $\mathrm{SU}(3) \times \mathrm{U}(1)$, the decomposition of representations of SO(12) now descends to:

$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1)_b \supset \mathrm{SO}(9) \times \mathrm{U}(1)_b \tag{C.575}$$

$$\supset [\mathrm{SU}(2) \times \mathrm{SU}(4)] \times \mathrm{U}(1)_b \tag{C.576}$$

$$\supset [\mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)_a]] \times \mathrm{U}(1)_b \tag{C.577}$$

$$12 \to (1,1)_{0,2} + (1,1)_{0,-2} + (1,1)_{0,0} + (3,1)_{0,0} \tag{C.578}$$

$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} \tag{C.579}$$

$$32, 32' \to (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.580}$$
$$+ (2,1)_{-3,-1} + (2,\overline{3})_{1,-1} + (2,1)_{3,-1} + (2,3)_{-1,-1} \tag{C.581}$$
$$66 \to (1,1)_{0,0} + (1,1)_{0,2} + (1,1)_{0,-2} + (3,1)_{0,2} \tag{C.582}$$
$$+ (3,1)_{0,-2} + (3,1)_{0,0} + (1,3)_{2,2} + (1,\overline{3})_{-2,2} \tag{C.583}$$
$$+ (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} + (3,1)_{0,0} + (1,1)_{0,0} \tag{C.584}$$
$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} + (1,3)_{-4,0} + (1,\overline{3})_{4,0} \tag{C.585}$$
$$+ (1,8)_{0,0} + (3,3)_{2,0} + (3,\overline{3})_{-2,0}. \tag{C.586}$$

Listing all Q-, U- and H_u- fields which can yield the term QUH_u, we find:

	Q	U	H_u	(a,b)
1	$(2,3)_{-1,\pm1}$	$(1,\overline{3})_{-2,0}$	$(2,1)_{3,\mp1}$	$(2,\pm3)$
2	$(2,3)_{-1,\pm1}$	$(1,\overline{3})_{-2,\mp2}$	$(2,1)_{3,\pm1}$	$(1/2,\pm3/2)$
3	$(2,3)_{-1,\pm1}$	$(1,\overline{3})_{4,0}$	$(2,1)_{-3,\mp1}$	$(-1,0)$

. (C.587)

Because the candidate E-fields all descend from the representation $(1,1)_{0,\pm2}$, it follows that $b = \pm3$ so that the second and third cases are ruled out. Restricting to this case, the candidate D-fields are therefore $(1,\overline{3})_{-2,\pm2}$, where the $\pm$ sign is correlated with that given in the first case. In order to obtain a gauge invariant QDH_d interaction term, the resulting H_d-field must transform in the representation $(2,1)_{3,\mp3}$, which does not descend from a representation of SO(12). We therefore conclude that this breaking pattern cannot yield the spectrum of the MSSM.

SO(12) ⊃ SO(10) × U(1) ⊃ [SU(2) × SO(7)] × U(1)

In this final case, SU(4) and G_2 or the only maximal subgroups of SO(7) which contains an SU(3) subgroup. Of these two possibilities, an instanton can only break the former case to SU(3). Decomposing the representations of SO(12) under the corresponding nested sequence of maximal subgroups yields:

$$\mathrm{SO}(12) \supset \mathrm{SO}(10) \times \mathrm{U}(1)_v \supset [\mathrm{SU}(2) \times \mathrm{SO}(7)] \times \mathrm{U}(1)_b \supset [\mathrm{SU}(2) \times \mathrm{SU}(4)] \times \mathrm{U}(1)_b \tag{C.588}$$
$$\supset [\mathrm{SU}(2) \times [\mathrm{SU}(3) \times \mathrm{U}(1)_a]] \times \mathrm{U}(1)_b \tag{C.589}$$
$$12 \to (1,1)_{0,2} + (1,1)_{0,-2} + (1,1)_{0,0} + (3,1)_{0,0} \tag{C.590}$$
$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} \tag{C.591}$$
$$32, 32' \to (2,1)_{3,1} + (2,3)_{-1,1} + (2,1)_{-3,1} + (2,\overline{3})_{1,1} \tag{C.592}$$
$$+ (2,1)_{-3,-1} + (2,\overline{3})_{1,-1} + (2,1)_{3,-1} + (2,3)_{-1,-1} \tag{C.593}$$
$$66 \to (1,1)_{0,0} + (1,1)_{0,2} + (1,1)_{0,-2} + (3,1)_{0,2} \tag{C.594}$$
$$+ (3,1)_{0,-2} + (3,1)_{0,0} + (1,3)_{2,2} + (1,\overline{3})_{-2,2} \tag{C.595}$$
$$+ (1,3)_{2,-2} + (1,\overline{3})_{-2,-2} + (3,1)_{0,0} + (1,1)_{0,0} \tag{C.596}$$
$$+ (1,3)_{2,0} + (1,\overline{3})_{-2,0} + (1,3)_{-4,0} + (1,\overline{3})_{4,0} \tag{C.597}$$
$$+ (1,8)_{0,0} + (3,3)_{2,0} + (3,\overline{3})_{-2,0}. \tag{C.598}$$

In fact, this decomposition is identical to that given for the previously considered nested sequence of maximal subgroups described by lines (C.575)–(C.586). We therefore conclude that just as in that case, this breaking pattern cannot yield the spectrum of the MSSM.

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